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Doubly periodic non-homogeneous Poisson models for hurricane data

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Abstract

Non-homogeneous Poisson processes with periodic claim intensity rate have been proposed as claim counts in risk theory. Here a doubly periodic Poisson model with short- and long-term trends is studied. Beta-type intensity functions are presented as illustrations. The likelihood function and the maximum likelihood estimates of the model parameters are derived.

Doubly periodic Poisson models are appropriate when the seasonality does not repeat exactly the same short-term pattern every year, but has a peak intensity that varies over a longer period. This reflects periodic environments like those forming hurricanes, in alternating El Niño/La Niña years. An application of the model to the data set of Atlantic hurricanes affecting the United States (1899–2000) is discussed in detail.

Keywords: Non-homogeneous Poisson process; Claim intensity function; Periodicity; Doubly periodic Poisson model; Maximum likelihood estimation; Hurricanes; El Niño/La Niña

1. Introduction

Non-homogeneous Poisson (NHP) processes are considered a more realistic alternative than the classical Poisson process for modelling the frequency of claims in risk theory.

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The NHP time-dependent intensity function is appropriate for describing the fluctuations of risks, subject to seasonality in their claims intensity.

Beard et al. [3] and Daykin et al. [8] claim that the risk process is often subject to continual changes in risk propensity. This is true for both the long-term, systematic, slowly changing trends as well as the short-term random variations that affect the number of claims. The model to be employed must then suitably define a time-dependent function or a stochastic process $\{\lambda(t)\}_{t\geq 0}$, instead of the constant Poisson parameter λ.

Berg and Haberman [4] use a non-homogeneous Markov birth process, of which the NHP is a special case, to predict trends such as the time to the next claim or the expected total number of claims in a year in life insurance claim occurrences.

In practice, natural phenomena evolving in a periodic environment, or under seasonal conditions, affect insurance claims. For example, weather factors are known to affect car or fire insurance claims, while seasonal snow storms in the north and hurricanes or floods in the south affect property–casualty insurance. A periodic time-dependent intensity rate is a reasonable model for the claim frequency in such situations.

Chukova et al. [5] shows that a random variable X with almost-lack-of-memory

$$P\{X > x + c \mid X > c\} = P\{X > x\}, \qquad \text{for some } c,$$

has a periodic hazard rate (intensity) function of period c, $h_X(t) = \frac{f_X(t)}{\bar{F}_X(t)}$, for $t > 0$. Obvious applications in risk theory are in modelling random phenomena with seasonal effects; car accidents, hurricanes. Some characterization properties of the NHP process with periodic failure rate are derived in [5,9].

A compound NHP process with periodic claim intensity rate case, called a periodic risk model, is considered and the related ruin problems in these models are discussed by Dassios and Embrechts [7], Asmussen and Rolski [1,2] and Rolski et al. [18]. These use the theory of piecewise-deterministic Markov processes, together with some standard martingale techniques and a corresponding average arrival rate risk model, respectively.

Garrido et al. [10] exploit the corresponding properties in a risk model, where the claim intensity rates are modelled by a NHP process with (single) periodic intensity. Some properties of such processes, illustrated by a beta-shape periodic intensity function, are discussed. Morales [15] further explores the singly periodic NHP model by defining a Gaussian intensity with which he considers the problem of ruin through a simulation study.

Furthermore, Garrido and Lu [11] consider a model with a doubly periodic intensity rate, where periodicity does not repeat exactly the same pattern in each short-term period; rather, its peak intensity varies over a longer period. This model reflects periodic environments like those forming hurricanes, in alternating El Niño/La Niña years. Parametric forms of the doubly periodic intensity function, such as the double-beta and the sine-beta ones, are proposed. These parametric forms are fitted here to hurricane data, emphasizing the inferential aspects.

Tropical storms and hurricanes periodically affect every coastal US state along the Atlantic and the Gulf of Mexico, from Texas to Maine, year after year. According to Cole and Pfaff [6], much speculation exists regarding the significance of the El Niño effect. This is a phenomenon generating abnormally warm surface water temperatures off the coasts of Ecuador and Peru, affecting global climate in the short term, including weather patterns across North America. Particular attention has been directed toward the potential effects of the El Niño phenomenon on hurricane frequency and the strength attained by tropical cyclones during El Niño years, in comparison to non-El Niño years (called La Niña ones). These can be seen as long-term climatological and periodical effects on North American weather.

Parisi and Lund [17] study the annual arrival cycle and return period properties of landfalling Atlantic Basin hurricanes. A NHP process with a periodic intensity function is used to model the annual cycle of hurricane arrival times. The data used in their study contain all Atlantic Basin hurricanes that have made a landfall in the contiguous United States during the years 1935–98, inclusive. Kernel methods are used to estimate the intensity function and the standard normal kernel function is selected.

In this paper, apart from considering the seasonal effects on the hurricane arrival times, we also consider global climatological and periodic effects and try to model the occurrence times of Atlantic hurricanes using a doubly periodic NHP process. A double-beta-type intensity function is used in this parametric model and the Atlantic hurricanes affecting the United States 1899–2000 data set [16,13] is used to estimate the parameters in the model. By contrast to the method proposed by Parisi and Lund [17], a parametric statistical inference approach is used here to estimate the intensity function. Maximum likelihood estimators of model parameters for this data set are obtained.

A brief description of the hurricane data set is given in Section 2. NHP models with singly or doubly periodic intensity are introduced. The statistical inference of the model parameters is presented in Section 3. Finally, in Section 4 we discuss the fit of different models to the hurricane data and give some comments. The appendix contains some tables and remarks used in the goodness-of-fit assessment.

2. The hurricane data set and proposed models

The data used for our study come from Neumann et al. [16], who report on 155 hurricanes that crossed or passed immediately adjacent to the Unites States coastline (Texas to Maine), 1899 through 1992. Landreneau [13] contains 12 additional hurricanes for the years 1993 to 2000 and is obtained from the National Hurricane Center Web site. Henceforth we call this combined data set “the hurricane data”. Thus, over the 102-year period 1899 to 2000, a total of 167 category 1 to 5 hurricanes crossed the Atlantic United States coastline at one or more points.

The average annual number is 1.64 over the whole period, which means an average of one to two hurricane landfalls per year. The years with a maximum number of six hurricanes were 1916 and 1985, while 19 out of the 102 years had no hurricanes. It can be observed that the hurricane season starts in June and ends in November over those years. Furthermore, the hurricane season peak period lasts from mid-August to October, with September having had the most major hurricanes (38.9% of all hurricanes). Fig. 1 shows the annual distribution of those 167 Atlantic hurricanes, while Table 1 gives their monthly distribution (see Appendix A for a discussion on the fit of the doubly periodic model to this data set).

Let N_t, the number of events occurring in an interval of the form $[0, t)$, be a NHP process with intensity function $\lambda(t)$ for $t \geq 0$. By definition, the probability of n claims

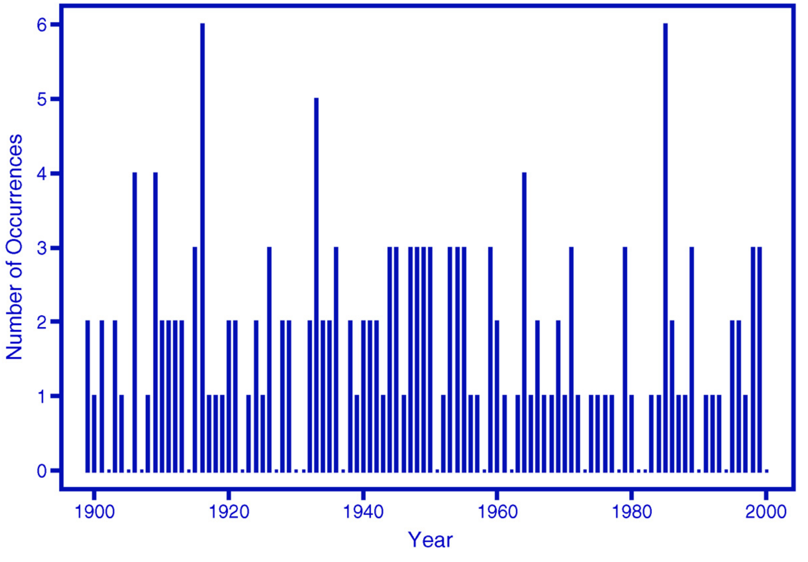

Fig. 1. US Atlantic hurricane (1899–2000) annual counts.

Table 1
Monthly distribution of the hurricane data

Month	Number of occurrences	Proportion (%)
June	11	6.6
July	17	10.2
August	44	26.3
September	65	38.9
October	26	15.6
November	4	2.4
	167	100.0

occurring in a time interval $[0, t)$ is given by

$$P\{N_t = n\} = \frac{e^{-\Lambda(t)}[\Lambda(t)]^n}{n!}, \quad n \in \mathbb{N}, \tag{1}$$

where Λ, called the cumulative hazard function or the cumulative intensity function of the process, is defined by $\Lambda(t) = \int_0^t \lambda(v)\mathrm{d}v$ for $t \geq 0$. That is, for a NHP process with intensity function λ, N_t has a Poisson distribution with mean $\Lambda(t)$.

When its intensity function does not depend on time, i.e. $\lambda(t) = \lambda$, for all $t \geq 0$, the corresponding NHP process is the classical homogeneous Poisson process, where $\Lambda(t) = \lambda t$ is linear. From Table 1 we see that here the maximum likelihood estimator (MLE) of λ would be $\hat{\lambda} = 1.64$ hurricanes per year, for the homogeneous Poisson model. See Appendix A for the goodness-of-fit analysis of this model.

Now, consider the case where the risk process evolves in a periodic environment, as when the claim arrival rate depends on the seasons. Then the intensity function of a NHP process is a periodic function, say with a period of $c > 0$ years. Consequently $t - \lfloor \frac{t}{c} \rfloor c \in [0, c)$, for $t \geq 0$, is the time of the season, where $\lfloor t \rfloor$ is the integer part of t.

Models with single and double periodicity are introduced in the following section, where they are illustrated with beta-type functions.

2.1. A singly periodic intensity model

Assume that the short-term period is 1 (year). Let λ_1 be a beta-type function, with parameters $p_1, q_1 \geq 1$, defined on $[0, 1]$, such that $\lambda_1(t_1^*) = 1$, where $t_1^* \in [0, 1]$ is the mode of the function. That is,

$$\lambda_1(t) = \begin{cases} \dfrac{\left(\frac{t-m_1}{D}\right)^{p_1-1}\left(1-\frac{t-m_1}{D}\right)^{q_1-1}}{\alpha_1^*}, & 0 \leq m_1 \leq t \leq m_2 \leq 1, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $D = m_2 - m_1$ and

$$\alpha_1^* = \left(\frac{t_1^* - m_1}{D}\right)^{p_1-1}\left(1 - \frac{t_1^* - m_1}{D}\right)^{q_1-1}, \tag{3}$$

is a scale factor, while

$$t_1^* = m_1 + D\frac{p_1 - 1}{p_1 + q_1 - 2}, \tag{4}$$

is the mode of λ_1, so at the mode, $\lambda_1(t_1^*) = 1$ is the peak level.

Then the singly periodic beta intensity function is given by

$$\lambda(t) = \lambda_0^* \lambda_1\left(t - \lfloor t \rfloor\right), \qquad \text{for } t \geq 0, \tag{5}$$

where $\lambda_0^* > 0$ is the (constant) peak level for this intensity and λ_1 is given in (2).

The corresponding cumulative intensity function $\Lambda(t)$ is

$$\Lambda(t) = \frac{\lambda_0^* D}{\alpha_1^*}\left[\lfloor t \rfloor B(p_1, q_1) + B\left(p_1, q_1; \frac{t - \lfloor t \rfloor - m_1}{D}\right)\right], \qquad t \geq 0, \tag{6}$$

where

$$B(p, q) = \int_0^1 v^{p-1}(1-v)^{q-1}\mathrm{d}v = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

is the beta function at $p, q > 0$, while

$$B(p, q; t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \displaystyle\int_0^t v^{p-1}(1-v)^{q-1}\,\mathrm{d}v, & \text{if } t \in (0, 1), \\ B(p, q), & \text{if } t \geq 1 \end{cases}$$

is the usual incomplete beta function.

Following the results of Garrido and Lu [11], the NHP process $\{N_t\}_{t\geq 0}$ with intensity function given in (5) can be decomposed as

$$N_t = M_1 + M_2 + \cdots + M_{\lfloor t\rfloor} + N_{t-\lfloor t\rfloor}, \qquad t > 0,$$

where $\{M_i\}_{i\geq 1}$ are i.i.d. Poisson random variables distributed as N_1, with mean $\Lambda(1)$, representing counts for complete years. These M_i are independent of $N_{t-\lfloor t\rfloor}$, the latter being a Poisson r.v. with mean $\Lambda(t-\lfloor t\rfloor)$, for $t-\lfloor t\rfloor \in [0,\ 1)$, representing the count in the final incomplete year. Here $\Lambda(1)$ and $\Lambda(t-\lfloor t\rfloor)$ can be derived from (6), respectively.

An alternative simple form for λ_1, which can result in a better fit with real data, is the generalized three-parameter beta function (denoted as $G3B(p_1, q_1, \epsilon)$; see [12], Chapter 25), given by

$$\lambda_1(t) = \begin{cases} \dfrac{\left(\frac{t-m_1}{D}\right)^{p_1-1}\left(1-\frac{t-m_1}{D}\right)^{q_1-1}}{\alpha_1^*\left[1-(1-\epsilon)\left(\frac{t-m_1}{D}\right)\right]^{p_1+q_1}}, & 0 \leq m_1 \leq t \leq m_2 \leq 1, \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

where $p_1, q_1 \geq 1, \epsilon > 0, D = m_2 - m_1$ and

$$\alpha_1^* = \frac{\left(\frac{t_1^*-m_1}{D}\right)^{p_1-1}\left(1-\frac{t_1^*-m_1}{D}\right)^{q_1-1}}{\left[1-(1-\epsilon)\frac{t_1^*-m_1}{D}\right]^{p_1+q_1}}, \tag{8}$$

is again a scale factor, while

$$t_1^* = m_1 + D\frac{3-p_1-(1+q_1)\epsilon+\sqrt{[1+p_1+(1+q_1)\epsilon\,]^2-8(p_1+q_1)\epsilon}}{4(1-\epsilon)}, \tag{9}$$

is the mode of function λ_1, given by (7), such that $\lambda_1(t_1^*) = 1$. Note that as $\epsilon \to 1$, (9) tends to (4).

Then for the intensity function, given by (5), the corresponding cumulative intensity function Λ is derived as

$$\Lambda(t) = \frac{\lambda_0^* D}{\alpha_1^* \epsilon^{p_1}}\left[\lfloor t\rfloor B(p_1, q_1) + B\left(p_1, q_1; \frac{\epsilon\left\{\frac{t-\lfloor t\rfloor - m_1}{D}\right\}}{1-(1-\epsilon)\left\{\frac{t-\lfloor t\rfloor-m_1}{D}\right\}}\right)\right], \quad t \geq 0. \tag{10}$$

2.2. A doubly periodic intensity model

Assume that the peak values or the levels of the short-term intensity function vary periodically with period c (an integer number of years). If, as above, the short-term intensity is the beta-shape function given in (2), then the doubly periodic beta intensity function is given by

$$\lambda(t)=\begin{cases}\lambda_0^*\lambda_1(t-\lfloor t\rfloor) & \text{if } 0\le t-\left\lfloor\frac{t}{c}\right\rfloor c<1\\ \lambda_1^*\lambda_1(t-\lfloor t\rfloor) & \text{if } 1\le t-\left\lfloor\frac{t}{c}\right\rfloor c<2,\\ \vdots & \vdots\\ \lambda_{c-1}^*\lambda_1(t-\lfloor t\rfloor) & \text{if } c-1\le t-\left\lfloor\frac{t}{c}\right\rfloor c<c\end{cases}\tag{11}$$

where $\lambda_0^*,\ldots,\lambda_{c-1}^*$ are all positive levels. The resulting cumulative intensity function $\Lambda(t)$ is given by

$$\begin{aligned}\Lambda(t)=&\left\lfloor\frac{t}{c}\right\rfloor DB(p_1,q_1)\sum_{j=0}^{c-1}\frac{\lambda_j^*}{\alpha_1^*}+DB(p_1,q_1)\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}\frac{\lambda_j^*}{\alpha_1^*}\\ &+DB\left(p_1,q_1;\frac{t-\lfloor t\rfloor-m_1}{D}\right)\frac{\lambda^*_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor}}{\alpha_1^*},\qquad t>m_1,\end{aligned}\tag{12}$$

and $\Lambda(t)=0$ for $0\le t\le m_1$.

The corresponding NHP process $\{N_t\}_{t\ge0}$ with doubly periodic intensity can be decomposed as

$$N_t=M_1+\cdots+M_{\lfloor\frac{t}{c}\rfloor}+N^*_{\frac{t-\lfloor t\rfloor-m_1}{D}},\qquad t\ge0,\tag{13}$$

where

$$N^*_{\frac{t-\lfloor t\rfloor-m_1}{D}}=\sum_{j=0}^{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1}N_c^{(j)}+N^{(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor)}_{\frac{t-\lfloor t\rfloor-m_1}{D}},\tag{14}$$

and the $\{M_i\}_{i\ge1}$ are i.i.d. Poisson distributed with mean $DB(p_1,q_1)\sum_{j=0}^{c-1}\frac{\lambda_j^*}{\alpha_1^*}$, while $N_c^{(j)}$ is Poisson with mean $DB(p_1,q_1)\frac{\lambda_j^*}{\alpha_1^*}$, for $j=0,1,\ldots,\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor-1$, respectively, and $N^{(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor)}_{\frac{t-\lfloor t\rfloor-m_1}{D}}$ is also Poisson with mean $DB(p_1,q_1;\frac{t-\lfloor t\rfloor-m_1}{D})\frac{\lambda^*_{\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor}}{\alpha_1^*}$. All these random variables are mutually independent.

2.3. A double-beta periodic intensity model

One way to reduce the number of free parameters, in the previous model in (11), is to assume a parametric form for the long-term intensity also. Here this is reasonable if it can be assumed that the short-term peak intensity values are affected periodically by some smoothly varying conditions, such as the surface water temperatures in El Niño/La Niña phenomenon.

More specifically, here we assume that the peak beta values, $\lambda_0^*,\ldots,\lambda_{c-1}^*$ in the short-term intensities, follow another continuous function of period c (an integer number of years), called the long-term intensity function. For instance, a beta function $\lambda_c(t)$ is also proposed for the long-term intensity:

$$\lambda_c(t) = a + \frac{b-a}{\alpha_c^*}\left(\frac{t-m_c}{c} - \left\lfloor\frac{t-m_c}{c}\right\rfloor\right)^{p_c-1} \times \left[1-\left(\frac{t-m_c}{c} - \left\lfloor\frac{t-m_c}{c}\right\rfloor\right)\right]^{q_c-1}, \qquad t > 0, \tag{15}$$

where

$$\alpha_c^* = \left(\frac{t_c^*-m_c}{c}\right)^{p_c-1}\left(1-\frac{t_c^*-m_c}{c}\right)^{q_c-1}, \tag{16}$$

is again a scale factor, so a and b are, respectively, the minimum and maximum amplitudes of the peak values. Here m_c is the starting point of the complete cycle of the long-term beta function and

$$t_c^* = m_c + c\left(\frac{p_c-1}{p_c+q_c-2}\right) \tag{17}$$

denotes the mode of λ_c.

Then the double-beta intensity function is given by

$$\lambda(t) = \lambda_c\left(\lfloor t\rfloor - \left\lfloor\frac{t}{c}\right\rfloor c + t_1^*\right)\lambda_1(t-\lfloor t\rfloor), \qquad \text{for } t \geq 0, \tag{18}$$

where λ_1 and λ_c are given in (2) and (15), respectively.

The solid line in Fig. 2 illustrates the shape of $\lambda(t)$ in (18), when $p_1 = 3$, $q_1 = 2$, $m_1 = \frac{5}{12}$, $D = \frac{6}{12}$, $c = 5$, $p_c = 2$, $q_c = 1\frac{2}{3}$, $m_c = 3.75$, $a = 3$ and $b = 7$. The peak values of the short-term beta λ_1 fall on the dotted line plotting the long-term beta λ_c. This serves to explain the fluctuations in the peak values of λ_1, the short-term beta periodicity.

If the intensity function λ is given by (18), then the corresponding cumulative intensity function Λ has the form

$$\Lambda(t) = \left\lfloor\frac{t}{c}\right\rfloor DB(p_1,q_1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*} + DB(p_1,q_1)\sum_{j=0}^{\lfloor t-\lfloor \frac{t}{c}\rfloor c\rfloor -1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*} + DB\left(p_1,q_1;\frac{t-\lfloor t\rfloor - m_1}{D}\right)\frac{\lambda_c\left(\left\lfloor t-\left\lfloor\frac{t}{c}\right\rfloor c\right\rfloor + t_1^*\right)}{\alpha_1^*}, \qquad t \geq m_1, \tag{19}$$

where $\lambda_c(t)$ is given by (15).

For any $t \geq 0$, the random variable N_t admits the same decomposition as in (13), where the $\{M_i\}_{i\geq 1}$ are i.i.d. Poisson, here with mean $DB(p_1,q_1)\sum_{j=0}^{c-1}\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$; they are independent of $N_c^{(j)}$, for $j = 0, 1, \ldots, \lfloor t - \lfloor\frac{t}{c}\rfloor c\rfloor - 1$, and of $N_{\frac{t-\lfloor t\rfloor - m_1}{D}}^{(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor)}$, which are also Poisson, here with means $DB(p_1,q_1)\frac{\lambda_c(j+t_1^*)}{\alpha_1^*}$, for $j = 0, 1, \ldots, \lfloor t - \lfloor\frac{t}{c}\rfloor c\rfloor - 1$ and $DB\left(p_1,q_1;\frac{t-\lfloor t\rfloor - m_1}{D}\right)\frac{\lambda_c(\lfloor t-\lfloor\frac{t}{c}\rfloor c\rfloor + t_1^*)}{\alpha_1^*}$, respectively.

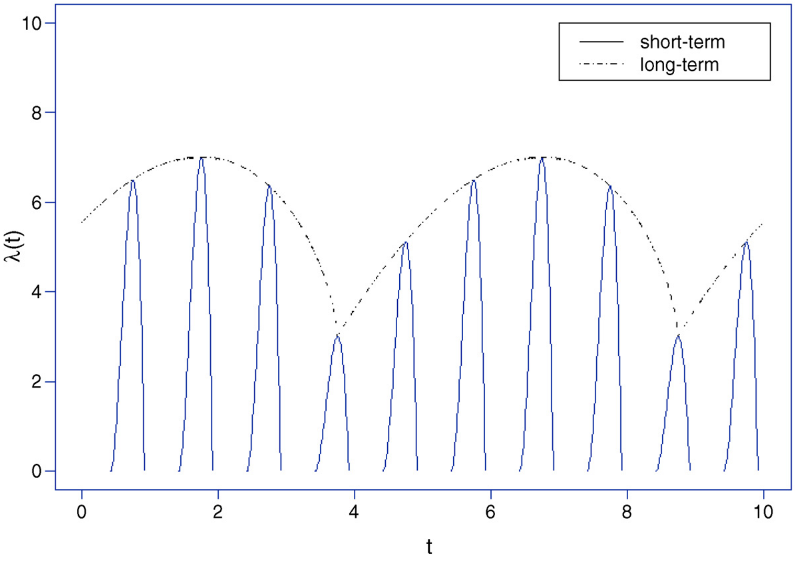

Fig. 2. The double-beta intensity function $\lambda(t)$.

This decomposition property of periodic NHP models is particularly useful for statistical inference, as seen in the following section.

3. Statistical inference

For the double-beta periodic intensity model in (18), the intensity is a parametric function with parameters p_1, q_1, p_c, q_c, a and b. It is possible to estimate these parameters from data using maximum likelihood estimation. Note that other model parameters such as m_1, m_2, m_c and c can usually be set at values observed from the data set.

Let d be the timescale in each short-term cycle; here $d = \frac{1}{12}$, a month in each year. Then, for the short-term intensity function in (2), denote by m_1 and m_2 two integer multiples of d; here m_1 and m_2 correspond to two specific months in the year, marking the beginning and end of the hurricane season. Furthermore, define J as

$$J = \frac{m_2 - m_1}{d} = \frac{D}{d},$$

that is the total number of months in each year over which the intensity function is positive. This gives a convenient partition of each year cycle $[0, m_1), [m_1, t_1), [t_1, t_2), \dots, [t_J, m_2), [m_2, 1]$, where

$$t_j = m_1 + jd, \qquad \text{for } j = 0, \dots, J. \tag{20}$$

Under the double-beta intensity function given in (18), the contribution to the likelihood for the first year of the first cycle is

$$L_{1,1} = \mathrm{e}^{-\int_0^{m_1} \lambda(v)\mathrm{d}v} \prod_{j=1}^{J} \left[\mathrm{e}^{-\int_{t_{j-1}}^{t_j} \lambda(v)\mathrm{d}v} \left(\int_{t_{j-1}}^{t_j} \lambda(v)\mathrm{d}v \right)^{n_{j,1}^{(1)}} \right] \mathrm{e}^{-\int_{m_2}^{1} \lambda(v)\mathrm{d}v} \tag{21}$$

$$= \mathrm{e}^{-\int_0^{1} \lambda(v)\mathrm{d}v} \prod_{j=1}^{J} \left(\int_{t_{j-1}}^{t_j} \lambda(v)\mathrm{d}v \right)^{n_{j,1}^{(1)}},$$

where $n_{j,1}^{(1)}$ is the number of events which occurred within the j-th month $[t_{j-1}, t_j)$ of the first year of the first cycle, for $j = 1, \ldots, J$. The first and the last terms in (21) represent the likelihood of having no hurricanes outside the time interval $[m_1, m_2]$.

In general, the contribution to the likelihood by the k-th year of the i-th cycle is similarly given by

$$L_{k,i} = \mathrm{e}^{-\int_{k-1}^{k} \lambda(v)\mathrm{d}v} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\mathrm{d}v \right)^{n_{j,k}^{(i)}}, \; k = 1, \ldots, c, i = 1, \ldots, \left\lfloor \frac{t}{c} \right\rfloor,$$

where $n_{j,k}^{(i)}$ is the number of hurricanes within the j-th month of the k-th year of the i-th cycle.

Hence for the i-th cycle, $i = 1, \ldots, \lfloor \frac{t}{c} \rfloor$, the total contribution to the likelihood is given by

$$L_i = \prod_{k=1}^{c} L_{k,i} = \mathrm{e}^{-\int_0^{c} \lambda(u)\mathrm{d}v} \prod_{k=1}^{c} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\mathrm{d}v \right)^{n_{j,k}^{(i)}},$$

while the likelihood function for all $\lfloor \frac{t}{c} \rfloor$ complete cycles is

$$L_{\mathrm{comp}} = \mathrm{e}^{-\lfloor \frac{t}{c} \rfloor \int_0^{c} \lambda(v)\mathrm{d}v} \prod_{k=1}^{c} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\mathrm{d}v \right)^{\sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)}}. \tag{22}$$

Finally, the contribution to the likelihood from the last incomplete cycle is composed of the contributions from complete years in the last cycle, the complete months in the last incomplete year and the last incomplete month. For simplicity, setting $\tau_c = \lfloor t - \lfloor \frac{t}{c} \rfloor c \rfloor$ to be the number of years in the last incomplete cycle, we have

$$L_{\mathrm{incomp}} = \prod_{k=1}^{\tau_c} \left[\mathrm{e}^{-\int_{k-1}^{k} \lambda(v)\mathrm{d}v} \prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\mathrm{d}v \right)^{n_{j,k}^{(\lfloor \frac{t}{c} \rfloor+1)}} \right]$$

$$\times\, \mathrm{e}^{-\int_{\lfloor \frac{t}{c} \rfloor c+\tau_c}^{\lfloor \frac{t}{c} \rfloor c+\tau_c+m_1} \lambda(v)\mathrm{d}v} \prod_{j=1}^{J^*} \left[\mathrm{e}^{-\int_{\tau_c+t_{j-1}}^{\tau_c+t_j} \lambda(v)\mathrm{d}v} \left(\int_{\tau_c+t_{j-1}}^{\tau_c+t_j} \lambda(v)\mathrm{d}v \right)^{n_{j,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor+1)}} \right]$$

$$\times\, \mathrm{e}^{-\int_{\lfloor \frac{t}{c}\rfloor c+\tau_c+t_{J^*}}^{t} \lambda(v)\mathrm{d}v} \left(\int_{\tau_c+t_{J^*}}^{t-\lfloor \frac{t}{c}\rfloor c} \lambda(v)\mathrm{d}v\right)^{n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c}\rfloor+1)}}, \tag{23}$$

where

$$J^* = \left\lfloor \frac{t-\left\lfloor \frac{t}{c}\right\rfloor c - \left\lfloor t-\left\lfloor \frac{t}{c}\right\rfloor c\right\rfloor - m_1}{d} \right\rfloor = \left\lfloor \frac{t-\left\lfloor \frac{t}{c}\right\rfloor c - \tau_c - m_1}{d}\right\rfloor$$

is the number of months in the last incomplete year (set to 0 when J^* is a negative integer). Note here that the last two lines reduce to

$$\mathrm{e}^{-\int_{\lfloor \frac{t}{c}\rfloor c+\tau_c}^{\lfloor \frac{t}{c}\rfloor c+\tau_c+t-\lfloor t\rfloor} \lambda(v)\mathrm{d}v} = \mathrm{e}^{-\int_{\lfloor \frac{t}{c}\rfloor c+\tau_c}^{t} \lambda(v)\mathrm{d}v}, \qquad \text{for } t-\lfloor t\rfloor \le m_1.$$

Hence the full likelihood function is given by (22) and (23) as

$$\begin{aligned} L &= L_{\mathrm{comp}} \cdot L_{\mathrm{incomp}} \\ &= \mathrm{e}^{-\Lambda(t)} \prod_{k=1}^{\tau_c}\prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\mathrm{d}v\right)^{\sum_{i=1}^{\lfloor \frac{t}{c}\rfloor+1} n_{j,k}^{(i)}} \\ &\quad \times \prod_{k=\tau_c+1}^{c}\prod_{j=1}^{J} \left(\int_{(k-1)+t_{j-1}}^{(k-1)+t_j} \lambda(v)\mathrm{d}v\right)^{\sum_{i=1}^{\lfloor \frac{t}{c}\rfloor} n_{j,k}^{(i)}} \\ &\quad \times \prod_{j=1}^{J^*} \left(\int_{\tau_c+t_{j-1}}^{\tau_c+t_j} \lambda(v)\mathrm{d}v\right)^{n_{j,\tau_c+1}^{(\lfloor \frac{t}{c}\rfloor+1)}} \left(\int_{\tau_c+t_{J^*}}^{t-\lfloor \frac{t}{c}\rfloor c} \lambda(v)\mathrm{d}v\right)^{n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c}\rfloor+1)}}. \end{aligned} \tag{24}$$

Substituting λ for the double-beta periodic intensity function in (18), the integrals in (24) can be represented as incomplete beta functions, yielding

$$\begin{aligned} L &= \mathrm{e}^{-\Lambda(t)} \prod_{k=1}^{\tau_c}\prod_{j=1}^{J} \left\{ \frac{\lambda_c(k-1+t_1^*)}{\alpha_1^*} D\left[B\left(p_1,q_1;\frac{jd}{D}\right)\right.\right. \\ &\quad \left.\left. -\; B\left(p_1,q_1;\frac{(j-1)d}{D}\right)\right]\right\}^{\sum_{i=1}^{\lfloor \frac{t}{c}\rfloor+1} n_{j,k}^{(i)}} \\ &\quad \times \prod_{k=\tau_c+1}^{c}\prod_{j=1}^{J} \left\{ \frac{\lambda_c(k-1+t_1^*)}{\alpha_1^*} D\left[B\left(p_1,q_1;\frac{jd}{D}\right)\right.\right. \\ &\quad \left.\left. -\; B\left(p_1,q_1;\frac{(j-1)d}{D}\right)\right]\right\}^{\sum_{i=1}^{\lfloor \frac{t}{c}\rfloor} n_{j,k}^{(i)}} \\ &\quad \times \prod_{j=1}^{J^*} \left\{ \frac{\lambda_c(\tau_c+t_1^*)}{\alpha_1^*} D\left[B\left(p_1,q_1;\frac{jd}{D}\right) - B\left(p_1,q_1;\frac{(j-1)d}{D}\right)\right]\right\}^{n_{j,\tau_c+1}^{(\lfloor \frac{t}{c}\rfloor+1)}} \end{aligned}$$

$$\times \left\{ \frac{\lambda_c(\tau_c + t_1^*)}{\alpha_1^*} D \left[B\left(p_1, q_1; \frac{t - \lfloor \frac{t}{c} \rfloor c - \tau_c - m_1}{D}\right) - B\left(p_1, q_1; \frac{J^* d}{D}\right) \right] \right\}^{n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)}}, \tag{25}$$

where the function λ_c is given in (15).

Denote by $N = \sum_{i,j,k} n_{j,k}^{(i)}$ the total number of occurrences, for $1 \le i \le \lfloor \frac{t}{c} \rfloor + 1$, $1 \le j \le J$ and $1 \le k \le c$. Further denote by

$$n_{\cdot,k}^{(\cdot)} = \sum_{j=1}^{J} \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)}, \qquad k = 1, 2, \ldots, c,$$

the total number of occurrences in the k-th year of all complete cycles, while $n_{\cdot,k}^{(\lfloor \frac{t}{c} \rfloor + 1)}$ stands for the count in the k-th year of the last incomplete cycle. Similarly

$$n_{j,\cdot}^{(\cdot)} = \sum_{k=1}^{c} \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)}, \qquad j = 1, 2, \ldots, J,$$

denotes the total number of occurrences in the j-th month of all complete cycles, while $n_{j,\cdot}^{(\lfloor \frac{t}{c} \rfloor + 1)}$ stands for the count in the j-th complete month of the last incomplete cycle. Consequently, the log likelihood function is given by

$$\begin{aligned} l = & -\Lambda(t) + N \log \frac{D}{\alpha_1^*} + \sum_{k=1}^{c} \left[n_{\cdot,k}^{(\cdot)} + n_{\cdot,k}^{(\lfloor \frac{t}{c} \rfloor + 1)} \right] \log \lambda_c(k - 1 + t_1^*) \\ & + \sum_{j=1}^{J} \left[n_{j,\cdot}^{(\cdot)} + n_{j,\cdot}^{(\lfloor \frac{t}{c} \rfloor + 1)} \right] \log \left[B\left(p_1, q_1; \frac{jd}{D}\right) - B\left(p_1, q_1; \frac{(j-1)d}{D}\right) \right] \\ & + n_{J^*+1,\tau_c+1}^{(\lfloor \frac{t}{c} \rfloor + 1)} \log \left[B\left(p_1, q_1; \frac{t - \lfloor \frac{t}{c} \rfloor c - \tau_c - m_1}{D}\right) - B\left(p_1, q_1; \frac{J^* d}{D}\right) \right]. \end{aligned} \tag{26}$$

The maximum likelihood estimators for p_1, q_1, p_c, q_c, a and b in the double-beta intensity function are obtained by maximizing l numerically.

Similarly, the maximum likelihood estimators for parameters in the model given in Section 2.2 with a generalized three-parameter beta short-term intensity function can be derived as follows.

To simplify expressions, let t be an integer number here. Assume that the short-term intensity function for the k-th year of a cycle is of the generalized three-parameter beta form in (7) with parameters $p_1^{(k)}$, $q_1^{(k)}$ and $\epsilon^{(k)}$ and λ_k^* is the peak value, where $k = 1, 2, \ldots, c$. For $1 \le k \le \tau_c$, the log likelihood function is given by

$$l_k = -\left(\left\lfloor \frac{t}{c} \right\rfloor + 1 \right) \frac{\lambda_k^* D B(p_1^{(k)}, q_1^{(k)})}{\alpha_1^{(k)} [\epsilon^{(k)}]^{p_1^{(k)}}} + \left[n_{\cdot,k}^{(\cdot)} + n_{\cdot,k}^{(\lfloor \frac{t}{c} \rfloor + 1)} \right] \log \left(\frac{\lambda_k^* D}{\alpha_1^{(k)} [\epsilon^{(k)}]^{p_1^{(k)}}} \right)$$

$$+ \sum_{j=1}^{J} \left[n_{j,k}^{(\cdot)} + n_{j,k}^{(\lfloor \frac{t}{c} \rfloor + 1)} \right] \log \left[B \left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)} \frac{jd}{D}}{1 - (1 - \epsilon^{(k)}) \frac{jd}{D}} \right) - B \left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)} \frac{(j-1)d}{D}}{1 - [1 - \epsilon^{(k)}] \frac{(j-1)d}{D}} \right) \right], \tag{27}$$

where $n_{j,k}^{(\cdot)} = \sum_{i=1}^{\lfloor \frac{t}{c} \rfloor} n_{j,k}^{(i)}$ is the total number of occurrences in the j-th month of the k-th year of all complete cycles and $\alpha_1^{(k)}$ is the scale factor of the k-th year of each cycle. For $\tau_c < k \le c$, we have

$$l_k = - \left\lfloor \frac{t}{c} \right\rfloor \frac{\lambda_k^* D B(p_1^{(k)}, q_1^{(k)})}{\alpha_1^{(k)} [\epsilon^{(k)}]^{p_1^{(k)}}} + n_{\cdot,k}^{(\cdot)} \log \left(\frac{\lambda_k^* D}{\alpha_1^{(k)} [\epsilon^{(k)}]^{p_1^{(k)}}} \right) + \sum_{j=1}^{J} n_{j,k}^{(\cdot)} \log \left[B \left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)} \frac{jd}{D}}{1 - (1 - \epsilon^{(k)}) \frac{jd}{D}} \right) - B \left(p_1^{(k)}, q_1^{(k)}; \frac{\epsilon^{(k)} \frac{(j-1)d}{D}}{1 - (1 - \epsilon^{(k)}) \frac{(j-1)d}{D}} \right) \right]. \tag{28}$$

4. Discussion and remarks

As outlined in Section 2, the illustrative data set used here comprises 167 hurricanes that made a landfall somewhere on the Atlantic United States coastline, over the 102-year period 1899 to 2000. These exhibit clear seasonal patterns. First, all hurricanes happened in the months from June to November. September generated more major hurricanes than any other month. On average, there were 1 to 2 hurricane landfalls per year over the whole period. A short-term (annual) periodic model thus seems appropriate.

First consider a NHP model with single periodicity. Fig. 3 gives the generalized three-parameter beta intensity described in (7), that was fitted to these annual hurricane frequencies. The parameter MLE's here are $\hat{p}_1 = 1.9198$, $\hat{q}_1 = 11.3050$, $\hat{\epsilon} = 0.1349$ and $\hat{\lambda}_0^* = 6.5145$, obtained with the Excel solver using the method described in Section 3.

The constant intensity $\lambda_1(t) = \hat{\lambda} = 1.64$, the homogeneous Poisson process MLE, is also shown graphically in Fig. 3 for comparison. Graphically, it is clear that the classical model gives here a crude representation of hurricane frequencies (this hypothesis is tested more formally in Appendix A).

Climatological studies suggest that the hurricane intensity does not repeat exactly the same short-term pattern every year. Rather, it slightly varies from year to year, as in alternating El Niño–La Niña cycles. For example, research on the tropical cyclones affecting the coast of Texas during El Niño/La Niña years of 1900–1996 shows that the highest percentage of all major hurricanes which have affected the coast of Texas occurred when El Niño was present for at least part of the given year (see [6]). Some actuaries also believe that El Niño/La Niña cycles in the Pacific affect tropical storm systems in the Atlantic.

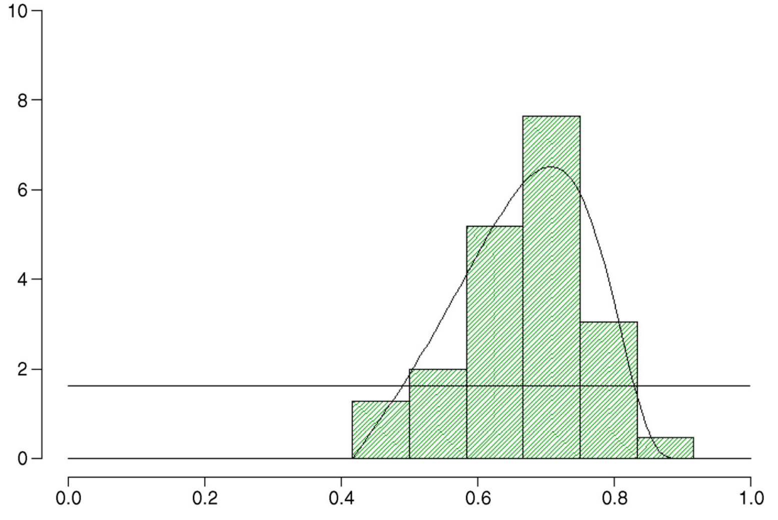

Fig. 3. The histogram and fitted hurricane intensity $\lambda(t)$ for 1-year cycles.

Our hurricane data also exhibit some long-term periodicity, under the influence of the global El Niño/La Niña phenomenon. The five-year cycle in Fig. 4 shows how the third and fourth years of the cycle have lower occurrences of hurricanes, the fourth year's being the lowest. This is followed by a peak lasting for a period of nearly three years (see Appendix A for a formal likelihood ratio goodness-of-fit test).

This motivates our assumptions of the doubly periodic NHP process presented in Section 2. Here the seasonality of the Atlantic hurricane repeats a similar short-term pattern every year; meanwhile the peak intensity, affected by the El Niño phenomenon, varies over a longer periodic cycle.

Climatologists observed that the typical El Niño cycle occurs within a 2–7 year cycle. From a graphical analysis of the data set, we conclude that a long-term period $c = 5$ years and a short-term period of one year describe the Atlantic hurricanes reasonably well.

Fig. 4 compares the observed and expected monthly average numbers of hurricanes over the five-year cycle for the 1899–2000 data set. A double two-parameter beta intensity function was used and the following MLE's were obtained: $\hat{p}_1 = 3.0145$, $\hat{q}_1 = 2.4389$, $\hat{p}_c = 1.5463$, $\hat{q}_c = 1.3642$, $\hat{a} = 3.2354$ and $\hat{b} = 6.9634$, where $\hat{q}_1$ and $\hat{q}_c$ are obtained from (4) and (17), respectively, while the estimated standard deviations for $\hat{p}_1$, $\hat{p}_c$, $\hat{a}$ and $\hat{b}$ are 0.3582, 0.7653, 0.7890 and 0.9126, respectively (see [14], Appendix A2 for a derivation).

Climatology suggests that the levels for the long-term cycle are governed by some underlying smoothly changing function, represented by the second beta function. The fit for each short-term cycle seems quite good, supporting our periodic theory. But the model does not adequately explain the short-term peaks over the long-term cycle. The El Niño/La Niña phenomenon is global, perhaps too complex to capture with such a simple parametric model.

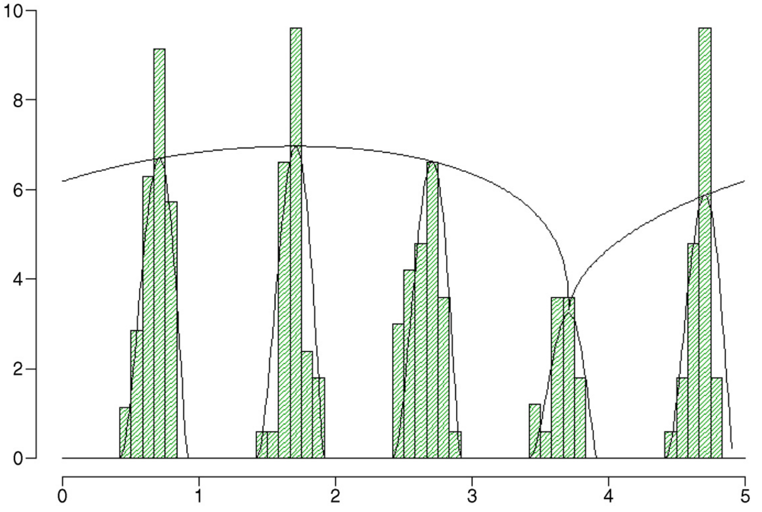

Fig. 4. Hurricane data and five-year double-beta intensities.

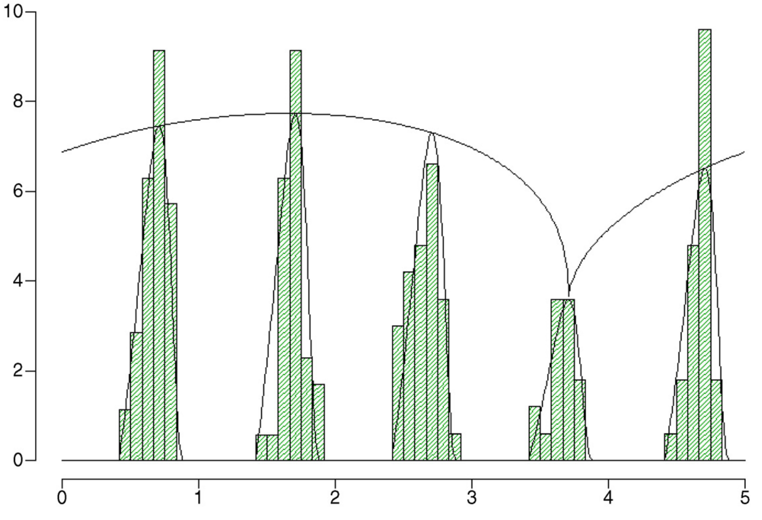

Fig. 5. Hurricane data and five-year generalized double-beta intensities.

Depending on the intended use of the model, the fit can be improved by the introduction of additional parameters. For instance when a generalized three-parameter beta intensity is used for the short-term cycle, while the long-term beta function is kept at two parameters, the following MLE's are obtained: $\hat{p}_1 = 1.8946$, $\hat{q}_1 = 12.3899$, $\hat{\epsilon} = 0.1205$, $\hat{p}_c = 1.5639$, $\hat{q}_c = 1.3921$, $\hat{a} = 3.5868$ and $\hat{b} = 7.7307$. It is clear from Fig. 5 that the fit is improved (although still not perfect), at the cost of introducing only one additional parameter.

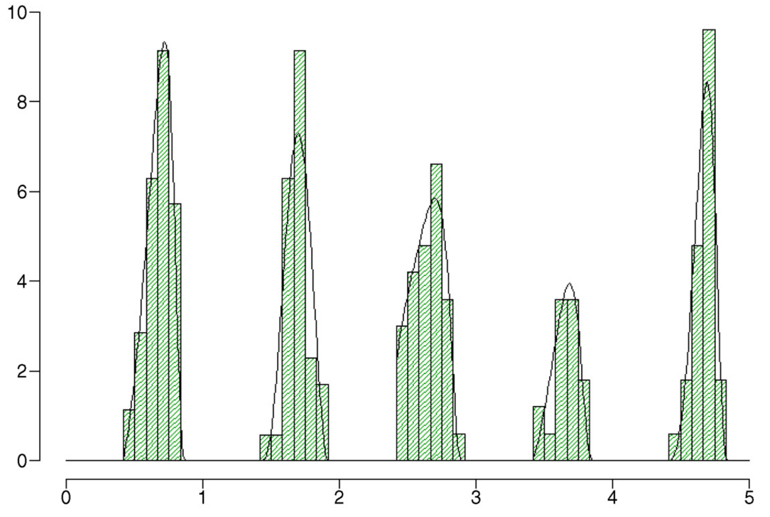

Fig. 6. Hurricane data five-year short-term generalized beta fit.

Table 2
MLE's of three-parameter beta intensities for the hurricane data

Year	$p_1^{(k)}$	$q_1^{(k)}$	$\epsilon^{(k)}$	λ_k^*
1	2.0087	150.0076	0.0097	9.3381
2	4.7926	3.0123	1.3227	7.2847
3	1.1459	11.9872	0.0820	5.8373
4	2.0586	121.7060	0.0150	3.9563
5	3.0769	155.4399	0.0165	8.4431

If fit is more important than simplicity of the model or smoothness, the number of parameters can be further increased by letting the short-term cycle peak values be free. Fig. 6 gives the histogram and fitted beta intensities, as in (11), for monthly hurricane frequencies over a five-year long-term cycle.

Here the generalized three-parameter beta function in (7) was used as the short-term intensity and the MLE's, given in Table 2, were derived from (27). The fit improvement is substantial for each short-term intensity in the five-year cycle. Yet, the model now fails to explain how hurricane intensities vary from El Niño to La Niña years. A possible remedy is the use of random effects on certain years of the cycle.

In conclusion, it appears that NHP risk models are more realistic in practice than classical Poisson processes, as their intensity rate is a function of time. This is clearly the case for hurricane landfalls.

In general, NHP processes with a periodic claim intensity can be useful in modelling risk processes that evolve in a periodic environment. The proposed double-beta periodic claim intensity not only generalizes the classical risk model, but also can give a more realistic

Table 3
Chi-squared goodness-of-fit testing for the homogeneous Poisson model

Counts	Observed	Expected	Chi squared
0	19	19.84	0.04
1	34	32.48	0.07
2	25	26.59	0.10
3	18	14.51	0.84
4+	6	8.57	0.77
Total	102	102.00	1.81

representation than (singly) periodic models with only short-term periodic intensity functions.

The flexibility in shape of the beta function and the explicit results obtained for the risk process, as well as the tractability of the statistical estimation of model parameters, should make these double-beta periodic models easy to use in practice. We hope that the illustration of the hurricane data set serves to show that NHP risk models can also be tractable if properly parametrized.

Acknowledgments

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Appendix A. Goodness-of-fit analysis

Figs. 1 and 3 provide graphical evidence that annual and monthly, respectively, hurricane counts show a periodic behaviour.

More formally, we can test the alternate hypothesis of a constant hurricane intensity, $\lambda_1(t) = \hat{\lambda} = 1.637254902$, resulting in a Poisson number of hurricanes per year. Table 3 reports the Poisson expected and observed numbers of years with 0, 1, 2, 3 and 4 or more hurricanes (the last observations were grouped to be representative).

A simple chi-squared test ($X^2 = 1.81 < \chi^2_{3;\,0.05} = 7.81$) does not reject the homogeneous Poisson assumption. Still, it is clear from Table 3 that the fit is poor in the tail of the distribution.

The Poisson model with constant intensity predicts well the expected numbers of years with lower hurricane frequencies (e.g. $n = 0$, 1 or 2 hurricanes per year), but gives a poorer prediction of the numbers of years with higher frequencies ($n = 3$ and $n \geq 4$). The fit in the tail is usually very important in insurance applications.

Furthermore, the homogeneous Poisson model fails to recognize the short-term seasonal and long-term cyclical patterns that the hurricane data exhibit in Fig. 5. A more appropriate statistical indicator here is to test the significance of the additional parameters in our double-beta periodic models.

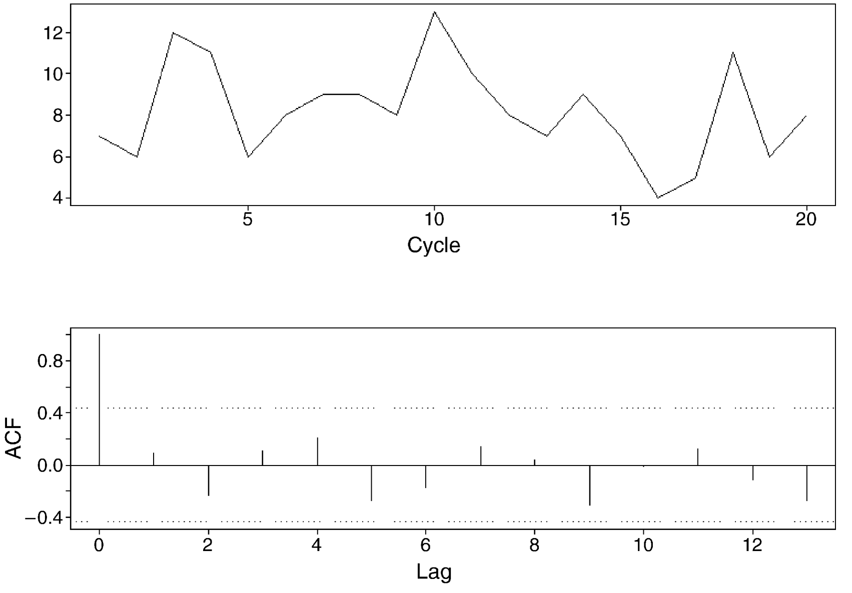

Fig. 7. Hurricane data autocorrelations; five-year cycle counts.

Since the classical Poisson model is a special case of the double-beta periodic model with four parameters, in Fig. 4, we call it the *reduced model*. A likelihood ratio test can be used to test the homogeneous Poisson hypothesis (reduced model), against the alternative of our more complete four-parameter model. The test statistic $r = 2(499.645 - 345.407) = 308.476 > \chi^2_{3;0.05} = 7.81$ is very significant, supporting the full *complete model* hypothesis.

Similarly, in testing for the extra parameter in our even more complete *full* model used for Fig. 5, with a generalized three-parameter beta function for the short-term intensity, the test statistic $r = 2(345.407 - 335.936) = 18.942 > \chi^2_{1;\,0.05} = 3.84$ is also very significant. This full double-beta periodic model with five parameters explains the observed periodicity more adequately than the above reduced and complete models.

The other assumption that should be tested is that of dependence on time. The hurricane counts observed here are not assumed to be mutually dependent (autocorrelated), but rather dependent on the time (season) of occurrence. Once a cycle completes, every five years, then this dependence on time gets reset. Subsequent five-year cycles are thus independent, as in the decomposition in (13). Fig. 7 shows the absence of autocorrelations, in these five-year cycle counts.

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Regime-Switching Periodic Models for Claim Counts

Abstract

We study a Cox risk model that accounts for both, seasonal variations and random fluctuations in the claims intensity. This occurs with natural phenomena that evolve in a seasonal environment and affect insurance claims, such as hurricanes.

More precisely, we define an intensity process, governed by a periodic function with a random peak level. The periodic intensity function follows a deterministic pattern in each short–term period, and is illustrated by a beta–type function. A two–state Markov chain defines the level process, explaining the random effect due to "high" or "low risk" years. This yields a regime–switching process, alternating between the two resulting intensities.

The properties of the corresponding claim counting process are discussed in detail. By properly defining the Lundberg coefficient, Lundberg–type bounds for finite time ruin probabilities are derived.

1 Introduction

Consider the risk process

$$U(t) = u + ct - \sum_{j=1}^{N(t)} X_j \; , \qquad t \geq 0 \; , \tag{1}$$

where u is the initial value, c is the (constant) premium rate, $\{N(t)\}_{t\geq 0}$ is a point process which models the number of claims arriving within the time interval $[0, t)$ and X_j is the j-th claim size. When $\{N(t)\}_{t\geq 0}$ is a Poisson process with (constant) intensity λ and the claim size sequence $\{X_j\}_{j\geq 1}$ are i.i.d. and independent of N, then (1) is known as the classical (homogeneous Poisson) risk model, which has been investigated extensively in the actuarial literature.

The classical risk model is not realistic in some practical situations. Two main modifications are made here. First, a non–homogeneous Poisson (NHP) process is used to model "size fluctuations" in the claim intensity of a risk subject to seasonality. Then, a Cox process, also called doubly stochastic Poisson process and a natural extension of the NHP process, is used to characterize the underlying "risk fluctuations" in the claims intensity [see Grandell (1991)].

The risk theory literature gives only a few results when the claim counting process is a NHP process. Dassios and Embrechts (1989) defines a risk model with periodic claim intensity and consider the corresponding ruin problems using a martingale approach. Similar models are also considered by Asmussen and Rolski (1994) and by Rolski et al. (1999). Two–sided bounds and asymptotic formulae for ruin probabilities are derived by using an average arrival rate risk model. Berg and Haberman (1994) uses a non–homogeneous Markov birth process, of which the NHP process is a special case, to predict trends in life insurance claim occurrences. Dimitrov et al. (2000) exploits some properties in a NHP risk model with a (single) periodic claim intensity. Morales (2004) chooses a Gaussian shape for the periodic claim intensity function. By contrast, Garrido and Lu (2002) considers a more general double periodic intensity rate. Possible forms of intensity functions, like the double–beta and the sine–beta, are proposed.

An early reference to Cox risk models is Ammeter (1948). In his model, the intensity λ_k over time intervals $[(k-1)\Lambda, k\Lambda)$ of (fixed) length Λ, for $k \in \mathbb{N}^+$, forms an i.i.d. sequence $\{\lambda_k\}_{k\geq 0}$. This model is generalized by Björk and Grandell (1988), who consider the intensity as $\lambda(t) = L_i$ if $\Sigma_{i-1} \leq t < \Sigma_i$, where $\Sigma_i = \sigma_1 + \cdots + \sigma_i$, with $\Sigma_0 = 0$ and (L_i, σ_i) a sequence of i.i.d. random vectors. Ammeter's model is revisited by Grandell (1995) and more properties of the model are discovered.

Asmussen (1989) proposes a Cox risk model, called a Markov–modulated Poisson process, whose intensity process $\{\lambda(t)\}_{t\geq 0}$ is given by $\lambda(t) = \lambda_{J(t)}$. Here the process $\{J(t)\}_{t\geq 0}$ models the random environment of an insurance business and is assumed to be an irreducible continuous time Markov chain, with finite state space $\{1, 2, \ldots, l\}$. Furthermore, a Cox risk process with a piecewise constant intensity is considered by Schmidli (1996), where the sequence $\{L_i\}_{i \geq 1}$ of successive levels of the intensity forms a Markov chain.

Ruin probabilities have been studied in these Cox models with a piecewise constant intensity. Lundberg inequalities hold, provided some assumptions are fulfilled. These may not be practical due to the difficulty in estimating the Lundberg coefficient and evaluating some constants within the inequalities. Other papers regarding to this topic are Embrechts et al. (1993) and Schmidli (1997).

There are very few results in the risk theory literature regarding Cox processes with other than piecewise constant intensities. Recently, Schmidli (2003) considered a NHP model with doubly stochastic occurrences for the PCS catastrophes index, based on individual indices for PCS options, where the intensity is of the form $\Lambda\lambda(t)$, with Λ is stochastic and $\lambda(t)$ is a given function.

Some natural phenomena evolve in a seasonal environment subject to random fluctuations which, in turn, affect insurance claims. For example, tropical storms and hurricanes periodically affect the coastal US states along the Atlantic and the Gulf of Mexico. The claim intensity then forms a specific pattern for each year which can be modeled by a periodic function. Speculation exists regarding the significance and potential effects of the El Niño phenomenon on hurricane frequency and the strength attained by tropical cyclones during alternating El Niño/La Niña years. These are random effects that, in some sense, affect the risk propensity or the peak level of the seasonal intensity, which can be modeled by a stochastic process.

In this paper we propose a Cox model that accounts for both, the seasonal variations and the random fluctuations in the claims intensity. Beard et al. (1984) and Daykin et al. (1994) suggest an intensity process λ as a composition of some factors, such as the normal trend, deviations from it and the short–term variations in risk propensity. Here we simply consider an intensity process with the following structure

$$\lambda(t) = \lambda_S(t)\mathbf{q}(t) , \qquad t \geq 0 , \tag{2}$$

where $\lambda_S(t)$ is the short–term intensity function and $\mathbf{q}(t)$ is a stochastic (level) process. The periodicity of the short–term intensity function is also considered, which takes into account those insurance claims affected by a periodic environment, like hurricanes or seasonal storms. A Markov chain with two states, corresponding to two different (high and low) levels, is chosen for the level process, yielding a so called regime–switching process. Under this intensity process, properties of the claim counting process and its corresponding risk process are studied in detail. By properly choosing the Lundberg coefficient, Lundberg–type upper bounds for finite time ruin probabilities are derived.

The paper is organized as follows. The model is defined in Section 2. Section 3 discusses the properties of the claim counting process. This gives a precise description of the model characteristics, such as the probabilities of recording k claims during the time interval $[0, t)$, for $t \geq 0$ and $k \in \mathbb{N}$, and the expectation of the integrated intensities in (2). In Section 4 we derive Lundberg–type upper bounds for finite time ruin probabilities and illustrate the results by some examples.

2 A Cox model with a regime–switching periodic intensity

Consider an intensity process $\{\lambda(t)\}_{t\geq 0}$ governed by a deterministic pattern in each short–term period, say a year, and a random effect on its peak level, that is the amplitude of the pattern. This fixed intensity pattern can be seen as the short-term periodicity, like in the NHP process. Assume we have two different risk levels; λ_0 which represents the risk under "low season" conditions, while the other, λ_1, represents the peak intensity under "high risk" years. In practice, such conditions can be slippery roads, foggy days, stormy weather, years affected by the El Niño phenomenon and so on.

Furthermore, assume that the intensity level modulates by an irreducible discrete time Markov process, $\kappa = \{\kappa_n\}_{n\geq 0}$, with finite state space $\{0,1\}$ and the transition probability matrix P, given by

$$P = \begin{pmatrix} 1-p_{01} & p_{01} \\ p_{10} & 1-p_{10} \end{pmatrix} . \tag{3}$$

Without loss of generality, we assume that the short–term period is 1. Let β be a function defined on $[0,1]$, such that $\beta(t^*) = 1$, where $t^* \in [0,1]$ is the mode of the function. Consider the intensity process λ, given by

$$\lambda(t) = \lambda_{\kappa_{\lfloor t\rfloor}}\, \beta(t - \lfloor t \rfloor)\ , \qquad t \geq 0\ . \tag{4}$$

This gives $\lambda(n+t^*) = \lambda_{\kappa_n}\,\beta(t^*) = \lambda_{\kappa_n}$ for $n \in \mathbb{N}$, that is, the peak of the function $\lambda(t)$ within the $(n+1)$-th year [i.e. $t \in [n, n+1)$] is λ_{κ_n}, which changes according to the Markov chain κ. As such, we call λ_{κ_n} the intensity level for year $n+1$.

In the sequel, we illustrate the annual common intensity pattern as a beta–type function with parameters $p \geq 1$ and $q \geq 1$, given by

$$\beta(t) = \alpha^*\, t^{p-1}\,(1-t)^{q-1}\ , \qquad 0 \leq t \leq 1\ , \tag{5}$$

where α^* is a scale factor, given by

$$\alpha^* = \frac{1}{(t^*)^{p-1}\,(1-t^*)^{q-1}} \quad \text{and} \quad t^* = \frac{p-1}{p+q-2} \tag{6}$$

is the mode of $\beta(t)$, $t \in [0,1]$. As such, note that at the mode $\beta(t^*) = 1$ is the peak level [see Figure 1]. Also denote the beta function in the usual way

$$B(p,q) = \int_0^1 v^{p-1}(1-v)^{q-1}dv = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad p, q \geq 1,$$

and the incomplete beta function at $p,\ q \geq 1$ as

$$B(p,q;t) = \int_0^t v^{p-1}\,(1-v)^{q-1}\,dv\ , \qquad t \in (0,\ 1)\ ,$$

with $B(p,q;t) = 0$ if $t \leq 0$, while $B(p,q;t) = \lfloor t \rfloor B(p,q) + B(p,q;t - \lfloor t \rfloor)$, if $t \geq 1$.

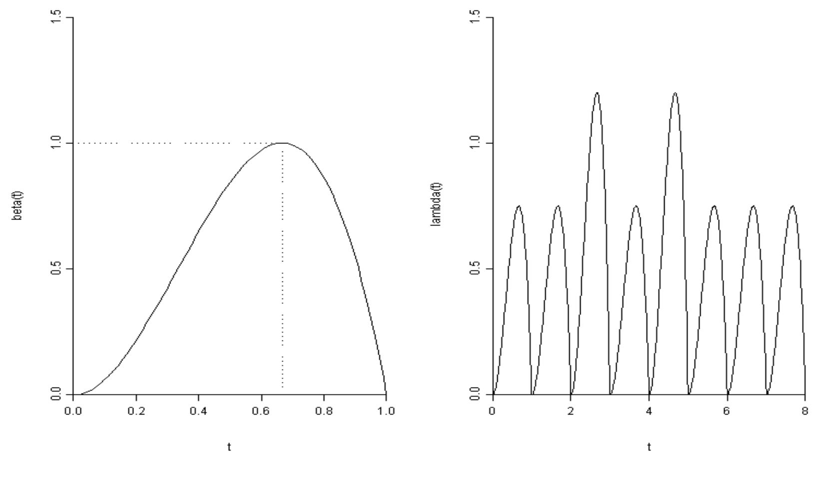

Figure 1: $\beta(t)$ and one realization of intensity process λ.

Figure 1 illustrates function $\beta(t)$, when $p = 3$ and $q = 2$, as well as a realization of the intensity process λ, when $p = 3$, $q = 2$, $\lambda_0 = 0.75$, $\lambda_1 = 1.2$, $p_{01} = 0.25$ and $p_{10} = 0.5$.

Consider a special Cox process, the claim counting process $\{N(t)\}_{t \geq 0}$ with an intensity process as in (4). Due to the periodicity of the function $\beta(t - \lfloor t \rfloor)$, for $t \geq 0$, and the transitions, from year to year, between levels λ_0 and λ_1, we call this risk model a regime–switching periodic non–homogeneous Poisson (NHP) process.

Let $\{N_i(t)\}_{t\geq 0}$ for $i = 0, 1$, (with $N_i(0) = 0$) denote a claim counting NHP process with intensity function $\lambda_i\, \beta(t - \lfloor t \rfloor)$ over the time interval $[0, t)$. That is $N_i(t)$ is Poisson distributed with mean $\lambda_i \int_0^t \beta(v - \lfloor v \rfloor)\, dv = \lambda_i\, \alpha^*\, B(p, q; t)$. Then the process $\{N(t)\}_{t\geq 0}$ can be represented as

$$N(t) = \sum_{i=0,1} Y_i(\lfloor t \rfloor)\, N_i(1) + N_{\kappa_{\lfloor t \rfloor}}(t) - N_{\kappa_{\lfloor t \rfloor}}(\lfloor t \rfloor)\ , \qquad t \geq 0\ , \tag{7}$$

where $Y_i(\lfloor t \rfloor) = \sum_{n=0}^{\lfloor t \rfloor - 1} \mathbb{I}(\kappa_n = i)$ denotes the number of years in $[0, \lfloor t \rfloor)$ that κ spends in state i, for $i = 0, 1$. This implies that, the conditional expected number of claims in the time interval $[0, t)$, given the environment, is:

$$\begin{aligned}
&\mathbb{E}\big[N(t) \mid \kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\big] \\
&\qquad = \sum_{n=0}^{\lfloor t \rfloor - 1} \int_0^1 \lambda_{\kappa_n}\, \beta(v)\, dv + \lambda_{\kappa_{\lfloor t \rfloor}} \int_0^{t - \lfloor t \rfloor} \beta(v)\, dv \\
&\qquad = L\big(\lfloor t \rfloor\big)\, \alpha^*\, B(p, q) + \lambda_{\kappa_{\lfloor t \rfloor}}\, \alpha^*\, B(p, q; t - \lfloor t \rfloor)\ , \qquad t \geq 0\ ,
\end{aligned}$$

where

$$L\big(\lfloor t \rfloor\big) = Y_0(\lfloor t \rfloor)\, \lambda_0 + Y_1(\lfloor t \rfloor)\, \lambda_1\ , \qquad t \geq 0\ , \tag{8}$$

denotes the sum of λ_0 and λ_1 values in $[0, \lfloor t \rfloor)$. Hence, we have

$$\mathbb{E}[N(t)] \leq \max\{\lambda_0, \lambda_1\}\, \alpha^*\, B(p, q; t).$$

The corresponding compound NHP process $\{S(t)\}_{t\geq 0}$ is given by

$$S(t) = \sum_{j=1}^{N(t)} X_j \ , \qquad t \geq 0 \ , \tag{9}$$

where the X_j's are the claim sizes with distribution function F_X, expected claim size $\mu = \int_0^\infty v \, dF_X(v)$ and moment generating function $\hat{m}_X(s) = \int_0^\infty e^{sv} \, dF_X(v)$, for some $s > 0$. These claim severities are assumed independent of the Markov environment process κ and hence of the claim counting process $\{N(t)\}_{t\geq 0}$. As in (7) process $\{S(t)\}_{t\geq 0}$ can also be represented as

$$S(t) \;=\; \sum_{i=0,1} Y_i(\lfloor t \rfloor) \, S_i(1) + S_{\kappa_{\lfloor t \rfloor}}(t) - S_{\kappa_{\lfloor t \rfloor}}(\lfloor t \rfloor) \ , \qquad t \geq 0 \ ,$$

where $S_i(t) = \sum_{j=1}^{N_i(t)} X_j$.

Now consider the continuous–time surplus process $\{U(t)\}_{t\geq 0}$, given by

$$U(t) = u + ct - S(t) \ , \qquad t \geq 0 \ , \tag{10}$$

where u is the initial capital value and c is the constant premium rate. The aggregate claim process $\{S(t)\}_{t\geq 0}$ is given in (9) and the claim counting process $\{N(t)\}_{t\geq 0}$ is the regime–switching periodic NHP process in (7).

Since the Markov environment process κ is assumed irreducible, it has a stationary initial distribution, denoted by $\pi = (\pi_0, \pi_1)$. Then by the law of large numbers for irreducible Markov processes, we have:

$$\lim_{t\to\infty} \frac{U(t)}{t} = c - \mu \sum_{i=0,1} \pi_i \, \lambda_i \, \alpha^* \, B(p,q) \ , \tag{11}$$

[see Rolski et al. (1999, Chapter 12)].

(11) implies that ruin occurs almost surely if the process has a negative drift, that is $c \leq \mu \sum_{i=0,1} \pi_i \, \lambda_i \, \alpha^* \, B(p,q)$. Therefore we assume that the net profit condition

$$c > \mu \sum_{i=0,1} \pi_i \, \lambda_i \, \alpha^* B(p,q) \ , \tag{12}$$

holds in the sequel.

3 Properties of the regime–switching periodic process

For the regime–switching periodic NHP process defined above, the random measure Λ in this Cox process, given the realization of the environment process κ up to time $\lfloor t \rfloor$, is:

$$\Lambda(t) = \int_0^t \lambda(v) \, dv = L\big(\lfloor t \rfloor\big) \, \alpha^* \, B(p,q) + \lambda_{\kappa_{\lfloor t \rfloor}} \, \alpha^* \, B(p,q; t - \lfloor t \rfloor) \ , \qquad t \geq 0 \ , \tag{13}$$

where $L\big(\lfloor t \rfloor\big)$ is given in (8). Then the conditional probability that the number of claims is k in the time interval $[0,t)$ is obtained as:

$$P\{N(t) = k \ | \ \kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\} = \frac{[\Lambda(t)]^k}{k!} \, e^{-\Lambda(t)} \ , \qquad k \in \mathbb{N}^+ \ ,$$

where $\Lambda(t)$ is given in (13).

In order to calculate $P\{N(t) = k\}$, we need to know how many times λ_0 appears in the sequence $\{\lambda_{\kappa_0}, \lambda_{\kappa_1}, \ldots, \lambda_{\kappa_{\lfloor t \rfloor}}\}$ (then the number of λ_1 values is fixed). This is equivalent to finding how many times 0 (say, "failure") or 1 ("success") appears in the corresponding sequence $\{\kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\}$. To do this, we denote $Y_i(n)$ to be the number of times that successive n-length sequences of the time–homogeneous $\{0, 1\}$-valued Markov process κ are in state i, for $i = 0, 1$.

Many papers discuss formulas or recursions for the distribution of success runs of several lengths in a two–state Markov chain [for example, see Han and Aki (1998)]. From these, it is not difficult to derive the distribution of the number of successes, $Y_1(n)$, which takes values in $\{0, 1, \ldots, n\}$ and can be obtained as follows.

Let $E_i(n, y)$ denote the conditional probability of y successes in a $(n + 1)$-length sequence, given that the sequence starts from state i, for $i = 0, 1$. That is, $E_0(n, y) = P\{Y_1(n) = y\}$ and $E_1(n, y) = P\{Y_1(n) = y - 1\}$. For convenience, define $E_i(n, y) = 0$ for all $y < 0$, $n \geq 0$ and $i = 0, 1$. We have the following recursive formulas for probabilities $E_i(n,y)$.

$$\begin{aligned}
E_i(0,0) &= 1\,, \qquad \text{for } i = 0,1, \\
E_0(n,y) &= (1-p_{01})\,E_0(n-1,y) + \sum_{m=1}^{n-1} p_{01}\,(1-p_{10})^{m-1} p_{10}\,E_0(n-m-1,y-m) \\
&\quad + p_{01}\,(1-p_{10})^{n-1}\,E_1(0,y-n)\,, \qquad \text{for } 0 \le y \le n,\ n \ge 1\,, \qquad (14) \\
E_1(n,y) &= p_{10}\,E_0(n-1,y) + \sum_{m=1}^{n-1} (1-p_{10})^{m} p_{10}\,E_0(n-m-1,y-m) \\
&\quad + (1-p_{10})^{n}\,E_1(0,y-n)\,, \qquad \text{for } 0 \le y \le n,\ n \ge 1.
\end{aligned}$$

Denote by $P_n(y)$, the probability of $Y_1(n) = y$ (implying that $Y_0(n)$ must be $n-y$) in a n-length sequence of the $\{0,1\}$-valued irreducible Markov chain κ. Then assuming that this n-length sequence starts κ_0, the law of the total probabilities gives:

$$\begin{aligned}
P_n(y) &= \sum_{i=0,1} P\{Y_1(n) = y \mid \kappa_0 = i\}\,P\{\kappa_0 = i\} \\
&= \pi_0\,P\{Y_1(n-1) = y \mid \kappa_0 = 0\} + \pi_1\,P\{Y_1(n-1) = y-1 \mid \kappa_0 = 1\} \\
&= \pi_0\,E_0(n-1,y) + \pi_1\,E_1(n-1,y-1) \\
&= \sum_{i=0,1} \pi_i\,E_i(n-1,y-i)\,, \qquad \text{for } 0 \le y \le n,\ n \in \mathbb{N}\,, \qquad (15)
\end{aligned}$$

where $E_i(n-1,y-i)$ can be recursively calculated from (14) and (π_0,π_1) is the initial distribution of Markov chain κ.

For example, in a 3-length sequence, the probability that there are no successes is given by:

$$P_3(0) = \pi_0\, E_0(2,0) = \pi_0\,(1-p_{01})\, E_0(1,0) = \pi_0\,(1-p_{01})^2\, E_0(0,0) = \pi_0\,(1-p_{01})^2\ ,$$

since first it has to start from state 0 and then stays at 0 (failure) for the next two steps. By contrast, the probability that there are 1 success and 2 failures is

$$\begin{aligned}
P_3(1) &= \pi_0\, E_0(2,1) + \pi_1\, E_1(2,0) \\
&= \pi_0\,\big[(1-p_{01})\, E_0(1,1) + p_{01}\, p_{10}\, E_0(0,0)\big] + \pi_1\, p_{10}\, E_0(1,0) \\
&= \pi_0\,\big[(1-p_{01})\, p_{01}\, E_1(0,0) + p_{01}\, p_{10}\big] + \pi_1\, p_{10}\,(1-p_{01})\, E_0(0,0) \\
&= \pi_0\,\big[(1-p_{01})\, p_{01} + p_{01}\, p_{10}\big] + \pi_1\, p_{10}\,(1-p_{01})\ ,
\end{aligned}$$

since if the sequence starts at 0, it must go to 1 once in the next two steps. But if the sequence starts at 1, it has to stay at 0 for the next two steps, and so on. Similarly, we have $P_3(2) = \pi_0\, p_{01}\,(1-p_{10}) + \pi_1\, p_{10}\,[p_{01} + (1-p_{10})]$ and $P_3(3) = \pi_1(1-p_{10})^2$.

We introduce the following notation for abbreviation. Denote by $\Lambda_n(y)$ the random measure under a realization of y periods at level λ_1 (and hence $n-y$ periods at λ_0), in the sequence $\{\lambda_{\kappa_0}, \lambda_{\kappa_1}, \ldots, \lambda_{\kappa_{n-1}}\}$. That is

$$\Lambda_n(y) = \big[(n-y)\lambda_0 + y\lambda_1\big]\, \alpha^*\, B(p,q)\ , \qquad 0 \le y \le n,\ n \in \mathbb{N}\ . \tag{16}$$

Then we have the following theorem for the probabilities $P\{N(t)=k\}$.

Theorem 1 Let $\kappa = \{\kappa_n\}_{n\geq 0}$ be a $\{0,1\}$-valued irreducible Markov chain with transition probabilities given by (3) and initial distribution (π_0, π_1). For the counting process $\{N(t)\}_{t\geq 0}$, given by (7), the probabilities that there be k claim occurrences during the time interval $[0,t)$, for $t \geq 0$ and $k \in \mathbb{N}$, is given by

$$\begin{aligned} P\{N(t)=k\} \quad = \quad & \sum_{y=0}^{\lfloor t \rfloor} P_{\lfloor t \rfloor}(y) \left\{ \sum_{i=0,1} (\pi_0\, p_{0i} + \pi_1\, p_{1i})\, e^{-\left[\Lambda_{\lfloor t \rfloor}(y) + \lambda_i\, \alpha^*\, B(p,q;t-\lfloor t \rfloor)\right]} \right. \\ & \left. \frac{\left[\Lambda_{\lfloor t \rfloor}(y) + \lambda_i\, \alpha^*\, B(p,q;t-\lfloor t \rfloor)\right]^k}{k!} \right\}, \end{aligned} \tag{17}$$

where $P_{\lfloor t \rfloor}(y)$ and $\Lambda_{\lfloor t \rfloor}(y)$ are obtained from (15) and (16), respectively.

Proof. See the Appendix. □

Note here that (17) can be re-written as

$$P\{N(t)=k\} = \mathbb{E}\big[P\{N(t)=k \,|\, \kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\}\big] = \mathbb{E}\left[\frac{\Lambda(t)^k}{k!}\, e^{-\Lambda(t)}\right],$$

where $\Lambda(t)$ is given by (13). It means that this regime–switching periodic NHP process can also be interpreted as a mixed Poisson process.

The random measure $\Lambda(t)$ of this special Cox process is given by (13). Taking expectations in (13) directly gives

$$\begin{aligned} \mathbb{E}[\Lambda(t)] \quad &= \quad \mathbb{E}\Big[L(\lfloor t \rfloor)\,\alpha^*\, B(p,q) + \lambda_{\kappa_{\lfloor t \rfloor}}\,\alpha^*\, B(p,q;t-\lfloor t \rfloor)\Big] \\ &= \quad \alpha^*\, B(p,q)\,\mathbb{E}\big[L(\lfloor t \rfloor)\big] + \alpha^*\, B(p,q;t-\lfloor t \rfloor)\,\mathbb{E}\big[\lambda_{\kappa_{\lfloor t \rfloor}}\big], \end{aligned}$$

then since $\mathbb{E}[Y_1(\lfloor t\rfloor)] = \sum_{y=0}^{\lfloor t\rfloor} y\, P_{\lfloor t\rfloor}(y)$ and $\mathbb{E}\big[\lambda_{\kappa_{\lfloor t\rfloor}}\big] = \lambda_0\big[\pi_0\,(1-p_{01}) + \pi_1\, p_{10}\big] + \lambda_1\big[\pi_0\, p_{01} + \pi_1\,(1-p_{10})\big]$, it follows that

$$\begin{aligned}\mathbb{E}\big[\Lambda(t)\big] &= \alpha^*\, B(p,q) \sum_{y=0}^{\lfloor t\rfloor} P_{\lfloor t\rfloor}(y)\,\big[(\lfloor t\rfloor - y)\,\lambda_0 + y\,\lambda_1\big] \\ &\qquad + \alpha^*\, B(p,q;t-\lfloor t\rfloor) \sum_{i=0,1} \lambda_i\,(\pi_0\, p_{0i} + \pi_1\, p_{1i})\;, \qquad t \geq 0\;. \quad (18)\end{aligned}$$

It is not difficult to see that (18) is equivalent to

$$\mathbb{E}[\Lambda(t)] = \sum_{y=0}^{\lfloor t\rfloor} P_{\lfloor t\rfloor}(y) \sum_{i=0,1} (\pi_0\, p_{0i} + \pi_1\, p_{1i})\,\big[\Lambda_{\lfloor t\rfloor}(y) + \lambda_i\, \alpha^*\, B(p,q;t-\lfloor t\rfloor)\big]\;,$$

then when $t \geq 0$ and $s < a_\Lambda$, the moment generating function of $\Lambda(t)$, $\hat{m}_{\Lambda(t)}(s) = \mathbb{E}[\exp\{s\Lambda(t)\}]$, is given by

$$\hat{m}_{\Lambda(t)}(s) = \sum_{y=0}^{\lfloor t\rfloor} P_{\lfloor t\rfloor}(y) \sum_{i=0,1} (\pi_0\, p_{0i} + \pi_1\, p_{1i})\, e^{s\left[\Lambda_{\lfloor t\rfloor}(y) + \lambda_i\, \alpha^*\, B(p,q;t-\lfloor t\rfloor)\right]}\;, \qquad (19)$$

where $a_\Lambda \in \mathbb{R}^+$ is such that $\lim_{s\uparrow a_\Lambda} \hat{m}_{\Lambda(t)}(s) = +\infty$, while $P_{\lfloor t\rfloor}(y)$ can be obtained from (15).

It is interesting to see that (19) can be re-written as

$$\begin{aligned}\hat{m}_{\Lambda(t)}(s) &= \sum_{y=0}^{\lfloor t\rfloor} P_{\lfloor t\rfloor}(y)\, e^{s\Lambda_{\lfloor t\rfloor}(y)} \sum_{i=0,1} (\pi_0\, p_{0i} + \pi_1\, p_{1i})\, e^{s\,\lambda_i\,\alpha^*\, B(p,q;t-\lfloor t\rfloor)} \\ &= \hat{m}_{\Lambda(\lfloor t\rfloor)}(s)\hat{m}_{\Lambda(t-\lfloor t\rfloor)}(s)\;, \qquad s < a_\Lambda\;,\end{aligned}$$

showing that $\Lambda(t) = \Lambda(\lfloor t\rfloor) + \Lambda(t-\lfloor t\rfloor)$ and that these are independent.

Theorem 1 and the above results on $\Lambda(t)$ allow for the derivation of the moments of $N(t)$. For instance, applying Fubini's Theorem and simple manipulations to

(17), gives the probability generating function $\hat{g}_{N(t)}(s) = \mathbb{E}[s^{N(t)}]$:

$$\begin{aligned} \hat{g}_{N(t)}(s) &= \sum_{y=0}^{\lfloor t \rfloor} P_{\lfloor t \rfloor}(y) \sum_{i=0,1} (\pi_0\, p_{0i} + \pi_1\, p_{1i})\, e^{(s-1)\,[\Lambda_{\lfloor t \rfloor}(y) + \lambda_i\, \alpha^*\, B(p,q;t-\lfloor t \rfloor)]} \\ &= \mathbb{E}[e^{(s-1)\,\Lambda(t)}] = \hat{m}_\Lambda(s-1)\ , \qquad |\, s \,| < 1\ . \end{aligned}$$

Furthermore, taking the r-th derivative of $\hat{g}_{N(t)}(s)$ with respect to $s \in (0,1)$, $\hat{g}^{(r)}_{N(t)}(s)$, and its limit as $s \uparrow 1$, yields the following successive factorial moments of $N(t)$ (that these be finite or not):

$$\begin{aligned} \mathbb{E}\Big[N(t)[N(t)-1]\cdots[N(t)-r+1]\Big] &= \hat{g}^{(r)}_{N(t)}(1) = \lim_{s\uparrow 1} \hat{g}^{(r)}_{N(t)}(s) \\ &= \lim_{s\uparrow 1} \mathbb{E}[\Lambda(t)^r\, e^{(s-1)\,\Lambda(t)}] = \mathbb{E}[\Lambda(t)^r]\ . \end{aligned}$$

In particular, we have that:

$$\mathbb{E}\big[N(t)\big] = \mathbb{E}\big[\Lambda(t)\big] \qquad \text{and} \qquad \mathbb{V}\big[N(t)\big] = \mathbb{V}\big[\Lambda(t)\big] + \mathbb{E}\big[\Lambda(t)\big]\ , \tag{20}$$

which imply that the index of dispersion of $N(t)$ is $\mathbb{I}_{N(t)} = \frac{\mathbb{V}[N(t)]}{\mathbb{E}[N(t)]} = 1 + \mathbb{I}_{\Lambda(t)} > 1$, showing that $N(t)$ is overdispersed, by contrast to the classical Poisson process.

4 A Lundberg upper bound for finite time ruin probabilities

This last section discusses the ruin problem for our special Cox process. The risk (income) process, over the time interval $[0,t)$, with initial value $R(0) = 0$ and a constant premium rate c, is given as

$$R(t) = ct - S(t) = ct - \sum_{j=1}^{N(t)} X_j \ , \qquad t \geq 0 \ , \tag{21}$$

where the claim counting process $\{N(t)\}_{t\geq 0}$ is the regime–switching periodic NHP process driven by our $\{0,1\}$-valued Markov chain κ and $S(t)$ is as in (9). Further assume that the moment generating function $\hat{m}_X(s) = \int_0^\infty e^{sx}\, dF_X(x)$ is twice differentiable on an interval $[0, a_X)$, where $a_X > 0$ and $\lim_{s\uparrow a_X} \hat{m}_X(s) = +\infty$.

Denote the Laplace–Stieltjes transform of $R(t)$ by $l(r;t) = \mathbb{E}[e^{-r\,R(t)}]$. Assuming it exists, it is given by

$$l(r;t) = e^{\Lambda(t)\left[\hat{m}_X(r)-1\right]-r\,ct} \ , \qquad r > a_{R(t)} \ , \ t \geq 0 \ . \tag{22}$$

Similarly, for $i = 0, 1$, let

$$\begin{aligned} l_i(r;t) &= \mathbb{E}\left[e^{-r\,R_i(t)}\right] = \mathbb{E}\left[e^{-r\,(ct-\sum_{j=1}^{N_i(t)} X_j)}\right] \\ &= e^{\lambda_i\,\alpha^*\,B(p,q;t)\,\left[\hat{m}_X(r)-1\right]-r\,ct} \ , \qquad r > a_{R_i(t)} \ , \ t \geq 0 \ . \end{aligned} \tag{23}$$

Let the time to ruin be defined in the usual way:

$$T_u = \inf\{t \geq 0 \ | \ u + R(t) < 0\} \ , \qquad u \geq 0 \ .$$

The ultimate ruin probability $\Psi(u)$ is then given by:

$$\Psi(u) = P\{T_u < \infty\} \ , \qquad u \geq 0 \ .$$

Using the martingale approach to Cox models discussed in Grandell (1991) we can prove the following result.

Theorem 2 The following Lundberg–type upper bound holds for the finite time ruin probability in model (21):

$$P\big\{T_u \leq t_0\big\} \leq e^{-ru}\,\mathbb{E}\left[\sup_{0\leq t\leq t_o} l(r;t)\right] , \qquad 0 \leq t_0 < \infty \,. \tag{24}$$

A tighter upper bound can also be obtained for $0 \leq t_0 < \infty$, as:

$$P\big\{T_u \leq t_0\big\} \leq e^{-ru}\,\mathbb{E}\left[\sup_{0\leq t\leq t_o} l(r;t)\right] \sup_{y\geq 0}\left\{\frac{e^{ry}\,\bar{F}_X(y)}{\int_y^{\infty} e^{rx}\,dF_X(x)}\right\} , \tag{25}$$

where $\bar{F}_X = 1 - F_X$ is the tail of the distribution function of X.

Proof. For details see the Appendix. $\square$

The upper bound given in (25) is difficult to use in practice. To derive a corresponding useful bound for our regime–switching periodic NHP model, first define the average risk level, given by

$$\bar{\lambda} = \pi_0\lambda_0 + \pi_1\lambda_1 \,, \tag{26}$$

and consider, for $r \geq 0$, the equation

$$\theta(r) = \bar{\lambda}\,\alpha^*\,B(p,q)\big[\hat{m}_X(r) - 1\big] - r\,c = 0 \,. \tag{27}$$

The solution, $\gamma > 0$, to (27) satisfies:

$$\bar{\lambda}\,\alpha^*\,B(p,q)\big[\hat{m}_X(\gamma) - 1\big] = \gamma c\ . \tag{28}$$

Here γ is an adjustment coefficient for the average risk level $\bar{\lambda}$ in (26), where λ_1, the peak intensity under "high risk" years, is assumed larger than that in the "low season" (i.e. $\lambda_0 < \lambda_1$). It follows from (28) that

$$\lambda_i\,\alpha^*\,B(p,q)\big[\hat{m}_X(\gamma) - 1\big] = \frac{\lambda_i}{\bar{\lambda}}\,\gamma\,c\ , \qquad i = 0, 1\ . \tag{29}$$

The existence and unicity of γ in $[0, a_X)$ is guaranteed because $\theta(0) = 0$ and $\theta'(0) = \bar{\lambda}\,\alpha^*\,B(p,q)\,\mu - c < 0$, provided that the net profit condition (12) holds, and hence the convexity of $\theta(r)$ ensures that $\theta'(\gamma) > 0$.

Assume that t_0 is an integer. Then with probabilities $P_{t_0}(y)$, given by (15), $\Lambda(t_0)$ takes the following realizations:

$$\Lambda_{t_0}(y) = \big[(t_0 - y)\,\lambda_0 + y\,\lambda_1\big]\,\alpha^*\,B(p,q)\ , \qquad 0 \le y \le t_0\ ,\ t_0 \in \mathbb{N}\ .$$

When $0 \le t \le t_0$, we have two possibilities for $\Lambda(t)$, depending on the value of $\lambda_{\kappa_{\lfloor t \rfloor}}$. One is

$$\Lambda(t) = \big[(\lfloor t \rfloor - z)\,\lambda_0 + z\,\lambda_1\big]\,\alpha^*\,B(p,q) + \lambda_0\,\alpha^*\,B(p,q; t - \lfloor t \rfloor)\ , \qquad 0 \le t \le t_0\ , \tag{30}$$

where $0 \le z \le \min\{\lfloor t \rfloor, y\}$ and $\lfloor t \rfloor - z + 1 \le t_0 - y$, or equivalently, $z \in C(t+1, y) =$

$\big[\max\{0, \lfloor t\rfloor + 1 - (t_0 - y)\}, \min\{\lfloor t\rfloor, y\}\big]$. While the other is

$$\Lambda(t) = \big[(\lfloor t\rfloor - z)\,\lambda_0 + z\,\lambda_1\big]\,\alpha^*\,B(p,q) + \lambda_1\,\alpha^*\,B(p,q;t-\lfloor t\rfloor)\ , \qquad 0 \le t \le t_0\ , \tag{31}$$

where similarly, $0 \le z \le \min\{\lfloor t\rfloor, y-1\}$ and $\lfloor t\rfloor - z \le t_0 - y$, or equivalently, $z \in C(t, y-1) = \big[\max\{0, \lfloor t\rfloor - (t_0 - y)\}, \min\{\lfloor t\rfloor, y-1\}\big]$.

When $\Lambda(t)$ is given by (30), then (28) and (29) imply that:

$$\begin{aligned}
\Lambda(t)\,\big[\hat{m}_X(\gamma) - 1\big] - \gamma\,ct &= (\lfloor t\rfloor - z)\,\big[\lambda_0\,\alpha^*\,B(p,q)\,(\hat{m}_X(\gamma)-1) - \gamma\,c\big] \\
&\quad + z\,\big[\lambda_1\,\alpha^*\,B(p,q)\,(\hat{m}_X(\gamma)-1) - \gamma\,c\big] \\
&\quad + \lambda_0\,\alpha^*\,B(p,q;t-\lfloor t\rfloor)\,(\hat{m}_X(\gamma)-1) - \gamma\,c(t-\lfloor t\rfloor) \\
&= -\lfloor t\rfloor\Big(\frac{\bar{\lambda}-\lambda_0}{\bar{\lambda}}\Big)\,\gamma\,c + z\,\big(\frac{\lambda_1-\lambda_0}{\bar{\lambda}}\big)\,\gamma\,c \\
&\quad + \lambda_0\,\alpha^*\,B(p,q;t-\lfloor t\rfloor)\,\big[\hat{m}_X(\gamma)-1\big] - \gamma\,c(t-\lfloor t\rfloor).
\end{aligned}$$

In turn

$$\begin{aligned}
\sup_{0\le t\le t_0} l(\gamma;t) &= \sup_{0\le t\le t_0} e^{\Lambda(t)\,\big[\hat{m}_X(\gamma)-1\big] - \gamma\,ct} \\
&\le \sup_{\substack{0\le t\le t_0\\ z\in C(t+1,y)}} e^{z\,\big(\frac{\lambda_1-\lambda_0}{\bar{\lambda}}\big)\gamma\,c + \lambda_0\,\alpha^*\,B(p,q;t-\lfloor t\rfloor)\,\big[\hat{m}_X(\gamma)-1\big] - \gamma\,c(t-\lfloor t\rfloor)} \\
&= \max_{\substack{0\le t\le t_0\\ z\in C(t+1,y)}} e^{z\,\big(\frac{\lambda_1-\lambda_0}{\bar{\lambda}}\big)\gamma\,c}\ \max_{0\le v<1} l_0(\gamma;v) \\
&\le e^{y\,\big(\frac{\lambda_1-\lambda_0}{\bar{\lambda}}\big)\gamma\,c}\ \max_{0\le v<1} l_0(\gamma;v)\ .
\end{aligned} \tag{32}$$

Similarly, when $\Lambda(t)$ is given by (31), then

$$\sup_{0\le t\le t_0} l(\gamma;t) \le e^{(y-1)\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c} \max_{0\le v<1} l_1(\gamma;v) \le e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c} \max_{0\le v<1} l_1(\gamma;v)\ ,$$

which has the same form as (32). Taking expectations gives

$$\mathbb{E}\left[\sup_{0\le t\le t_o} l(r;t)\right] = \sum_{y=0}^{t_0} P_{t_0}(y)\, e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c}\ .$$

Finally, setting $r=\gamma$, gives a Lundberg–type upper bound for the finite time ruin probability in (25), for $t_0\in\mathbb{N}$, that is:

$$P\{T_u\le t_0\} \le e^{-\gamma u}\left[\sum_{y=0}^{t_0} P_{t_0}(y)\, e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c}\right] \max_{\substack{0\le v<1\\ i=0,1}} l_i(\gamma;v)\ \sup_{y\ge 0}\left\{\frac{e^{ry}\,\bar{F}_X(y)}{\int_y^\infty e^{rx}\,dF_X(x)}\right\}\ , \tag{33}$$

where γ satisfies (28) and $P_{t_0}(y)$ is given in (15).

Obviously, the simpler bound for $P\{T_u\le t_0\}$ given by (24) can also be derived here:

$$P\{T_u\le t_0\} \le e^{-\gamma u}\left[\sum_{y=0}^{t_0} P_{t_0}(y)\, e^{y\left(\frac{\lambda_1-\lambda_0}{\lambda}\right)\gamma c}\right] \max_{\substack{0\le v<1\\ i=0,1}} l_i(\gamma;v)\ , \tag{34}$$

but (33) is tighter than (34), as shown in the following examples.

Example 1 Consider claim sizes that are exponentially distributed with mean μ. Their moment generating function $\hat{m}_X(s) = \frac{1}{1-\mu s}$, for $s < a_X = \frac{1}{\mu}$. The adjustment coefficient for parameter λ_0, is then given by

$$\gamma = \frac{c-\lambda_0\,\alpha^*\,B(p,q)\,\mu}{c\,\mu} = \frac{1}{\mu} - \frac{\bar{\lambda}\,\alpha^* B(p,q)}{c}\ , \tag{35}$$

which is the positive solution to equation (28). The corresponding $l_i(\gamma; v)$, given in (23), takes the form

$$l_i(\gamma; v) = e^{\left(\frac{\lambda_i}{\lambda}\frac{B(p,q;v)}{B(p,q)} - v\right)\gamma c}\,, \qquad 0 \le v < 1\,,\ i = 0, 1\,. \tag{36}$$

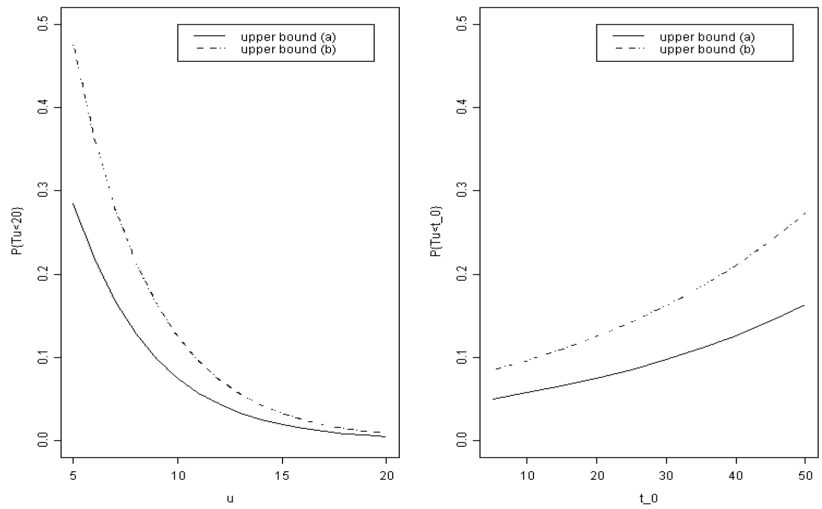

Figure 2: Upper bounds for exponential claims vs u ($t_0 = 20$) and t_0 ($u = 10$).

Figure 2 illustrates the upper bounds in this exponential case, as a function of u (left graph), when $t_0 = 20$, and as a function of t_0 (right graph), when $u = 10$. The other parameters are chosen to be $\lambda_0 = 1$, $\lambda_1 = 1.2$, $p = 3$, $q = 2$, $p_{01} = 0.25$, $p_{10} = 0.5$, $c = 1.5$, $\mu = 1.5$ and $\gamma = 0.267$, which is obtained from (35). Clearly, the upper bounds (a), given by (33) are sharper than those in (b), given by (34).

Example 2 Consider the case of inverse Gaussian distributed claims, with mean

μ, variance $\mu\,\beta$ and density function

$$f_X(x) = \frac{\mu}{\sqrt{2\pi\beta x^3}}\, e^{-\frac{(x-\mu)^2}{2\beta x}}\,, \qquad x > 0\,.$$

Their moment generating function $\hat{m}_X(s) = e^{\frac{\mu}{\beta}\,(1-\sqrt{1-2\beta s})}$ exists for $s < \frac{1}{2\beta}$. The adjustment coefficient γ with respect to parameter $\bar{\lambda}$ is the positive solution to the equation

$$\bar{\lambda}\,\alpha^*\,B(p,q)\left[e^{\frac{\mu}{\beta}\left(1-\sqrt{1-2\beta\,\gamma}\right)} - 1\right] = \gamma\,c\,, \tag{37}$$

and $l_i(v;\gamma)$, for $i = 0, 1$, is of the same form as in (36).

Figure 3 illustrates the upper bounds in this inverse Gaussian case, again as a function of u (left graph), when $t_0 = 20$, and as a function of t_0 (right graph), when $u = 10$. The other parameters are chosen as for Figure 2 and $\beta = \frac{8}{3}$, which gives a variance of 4. Here $\gamma = 0.155$ is obtained from (37). Again the upper bounds in (a), given by (33) are sharper than those in (b), given by (34).

Conclusions

Regime–switching periodic NHP processes can be useful in modeling risk processes under periodic and random environments. A beta–type short–term intensity function is proposed with a two–state Markov process to model the peak level in the

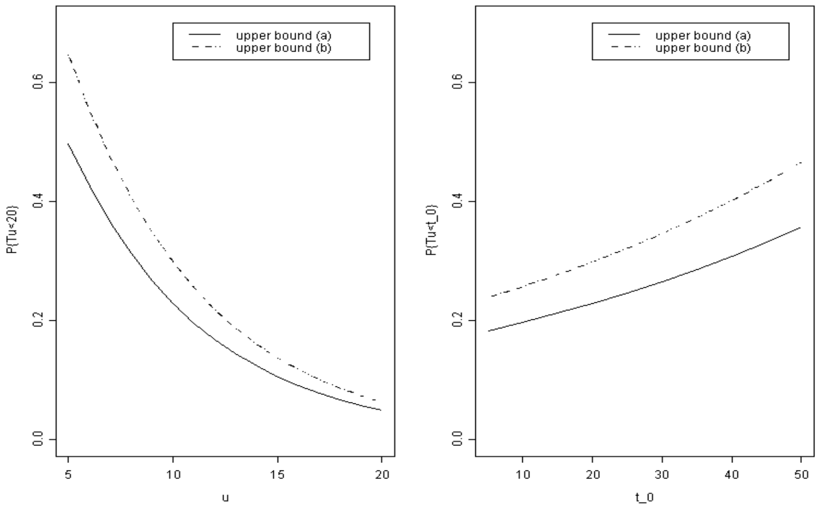

Figure 3: Upper bounds for inverse Gaussian claims vs t_0 ($u = 10$) and u ($t_0 = 20$).

intensity of this Cox risk process. This generalizes the periodic NHP model. It can also provide more realistic descriptions than Cox models with piecewise constant intensities.

The flexible shape of the beta function and the explicit results obtained for the Cox risk process should make these regime–switching period NHP models more practical than Cox processes with piecewise constant intensities, or than the usual NHP process. However, this work can be extended to other reasonable short–term intensity functions or regime–switching level processes with multi–state spaces. Furthermore, statistical methods to estimate from real data set the beta parameters and level parameters of the model are readily available and shall be illustrated in subsequent work.

Appendix

Proof of Theorem 1: By the law of the total probabilities, it is easily seen that

$$\begin{aligned} P\{N(t)=k\} &= P\{N(\lfloor t\rfloor)+[N(t)-N(\lfloor t\rfloor)]=k\} \\ &= \sum_{l=0}^{k} P\{N(\lfloor t\rfloor)=l\}\, P\{N(t)-N(\lfloor t\rfloor)=k-l\}\ . \end{aligned}$$

Furthermore, since

$$\begin{aligned} P\{N(\lfloor t\rfloor)=l\} &= \sum_{y=0}^{\lfloor t\rfloor} P\{N(\lfloor t\rfloor)=l \,\big|\, Y_1\lfloor t\rfloor)=y\}\, P\{Y_1(\lfloor t\rfloor)=y\} \\ &= \sum_{y=0}^{\lfloor t\rfloor} \frac{\big[\Lambda_{\lfloor t\rfloor}(y)\big]^l}{l!}\, e^{-\Lambda_{\lfloor t\rfloor}(y)}\, P_{\lfloor t\rfloor}(y)\ , \end{aligned}$$

and

$$\begin{aligned} & P\{N(t)-N(\lfloor t\rfloor)=k-l\} \\ &= \sum_{i=0,1} P\{N(t)-N(\lfloor t\rfloor)=k-l \,\big|\, \kappa_{\lfloor t\rfloor-1}=i\}\, P\{\kappa_{\lfloor t\rfloor-1}=i\} \\ &= \sum_{i=0,1} \Big[\sum_{j=0,1} P\{N(t)-N(\lfloor t\rfloor)=k-l \,\big|\, \kappa_{\lfloor t\rfloor-1}=i,\ \kappa_{\lfloor t\rfloor}=j\} \\ &\qquad\qquad P\{\kappa_{\lfloor t\rfloor}=j \,\big|\, \kappa_{\lfloor t\rfloor-1}=i\}\Big]\pi_i \\ &= \sum_{i=0,1} (\pi_0\, p_{0i}+\pi_1\, p_{1i})\, \frac{\big[\lambda_i\, \alpha^*\, B(p,q;t-\lfloor t\rfloor)\big]^{k-l}}{(k-l)!}\, e^{-\lambda_i\, \alpha^*\, B(p,q;t-\lfloor t\rfloor)}\ , \end{aligned}$$

we now can write

$$
\begin{aligned}
P\{N(t)=k\} &= \sum_{y=0}^{\lfloor t\rfloor} P_{\lfloor t\rfloor}(y) \Bigg\{ \sum_{i=0,1} (\pi_0\, p_{0i} + \pi_1\, p_{1i})\, e^{-\left[\Lambda_{\lfloor t\rfloor}(y)+\lambda_i\,\alpha^*\, B(p,q;t-\lfloor t\rfloor)\right]} \\
&\qquad \sum_{l=0}^{k} \frac{\left[\Lambda_{\lfloor t\rfloor}(y)\right]^l}{l!\,(k-l)!} \left[\lambda_i\,\alpha^*\, B(p,q;t-\lfloor t\rfloor)\right]^{k-l} \Bigg\} \\
&= \sum_{y=0}^{\lfloor t\rfloor} P_{\lfloor t\rfloor}(y) \Bigg\{ \sum_{i=0,1} (\pi_0\, p_{0i} + \pi_1\, p_{1i})\, e^{-\left[\Lambda_{\lfloor t\rfloor}(y)+\lambda_i\,\alpha^*\, B(p,q;t-\lfloor t\rfloor)\right]} \\
&\qquad \frac{\left[\Lambda_{\lfloor t\rfloor}(y) + \lambda_i\,\alpha^*\, B(p,q;t-\lfloor t\rfloor)\right]^k}{k!} \Bigg\} \ ,
\end{aligned}
$$

which completes the proof. □

Proof of Theorem 2: Consider the martingale approach to Cox models discussed in Grandell (1991). Let $\mathbf{F}$ be a suitable filtration, M be a positive $\mathbf{F}$-martingale (or a positive $\mathbf{F}$-supermartingale) and T be an $\mathbf{F}$-stopping time. Choose $t_0 < \infty$ and consider $t_0 \wedge T$, a bounded $\mathbf{F}$-stopping time.

By the optional stopping theorem, we have that

$$
M(0) \geq \mathbb{E}^{\mathcal{F}_0}[M(t_0 \wedge T)] \geq \mathbb{E}^{\mathcal{F}_0}[M(T) \mid t \leq t_0]\, P^{\mathcal{F}_0}\{T \leq t_0\} \ ,
$$

and therefore

$$
P^{\mathcal{F}_0}\{T \leq t_0\} \leq \frac{M(0)}{\mathbb{E}^{\mathcal{F}_0}[M(T) \mid T \leq t_0]} \ , \qquad t_0 < \infty \ .
$$

Let the risk process R be adapted to $\mathbf{F}$, that is $\mathcal{F}_t \supseteq \mathcal{F}_t^R$ for all $t \geq 0$. Then the ultimate ruin probability $\Psi(u)$ is seen to be:

$$
\Psi(u) = P\{T_u < \infty\} = \mathbb{E}\big[P^{\mathcal{F}_0}\{T_u < \infty\}\big] \ , \qquad u \geq 0 \ .
$$

Now consider N to be a Cox process with intensity process $\{\lambda(t)\}_{t\geq 0}$ and random intensity measure Λ, given by $\Lambda(t) = \int_0^t \lambda(v)\, dv$. A suitable filtration $\mathbf{F}$ is defined as $\mathcal{F}_t = \mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^R$ and thus $\mathcal{F}_0 = \mathcal{F}_\infty^\Lambda$. Consider the following choice of process M:

$$M(t) = \frac{e^{-r\,[u+R(t)]}}{l(r;t)} = \frac{e^{-r\,[u+R(t)]}}{e^{\Lambda(t)\,[\hat{m}_X(r)-1]-r\,ct}}\ , \qquad t \geq 0\ ,$$

where $R(t)$ is given in (21).

It can be shown that M is an $\mathbf{F}$-martingale where the filtration is given by $\mathcal{F}_t = \mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^R$. A lower bound is obtained when $0 \leq t_0 < \infty$ as

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_0}\big[M(T_u)\,\big|\,T_u \leq t_0\big] \ &\geq\ \mathbb{E}^{\mathcal{F}_0}\big[e^{-\Lambda(T_u)\,[\hat{m}_X(r)-1]+r\,cT_u}\,\big|\,T_u \leq t_0\big] \\ &\geq\ \inf_{0\leq t\leq t_o} e^{-\Lambda(t)\,[\hat{m}_X(r)-1]+r\,ct}\ . \end{aligned} \tag{38}$$

More precisely,

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_0}\big[M(T_u)\,\big|\,T_u \leq t_0\big] \\ &\quad = \mathbb{E}^{\mathcal{F}_0}\big[e^{-r\,[u+R(T_u)]}\,e^{-\Lambda(T_u)\,[\hat{m}_X(r)-1]+r\,cT_u}\,\big|\,T_u \leq t_0\big] \\ &\quad \geq \inf_{0\leq t\leq t_o}\big\{e^{-\Lambda(t)\,[\hat{m}_X(r)-1]+r\,ct}\big\}\,\mathbb{E}^{\mathcal{F}_0}\big[e^{-r\,[u+R(T_u)]}\,\big|\,T_u \leq t_0\big] \\ &\quad \geq \inf_{0\leq t\leq t_o}\big\{e^{-\Lambda(t)\,[\hat{m}_X(r)-1]+r\,ct}\big\}\,\inf_{y\geq 0}\left\{\frac{1-F_X(y)}{\int_y^\infty e^{-r(y-x)}\,dF_X(x)}\right\}\ . \end{aligned} \tag{39}$$

Then we get, from (38), that

$$P^{\mathcal{F}_0}\big\{T_u \leq t_0\big\} \leq \frac{M(0)}{\mathbb{E}^{\mathcal{F}_0}\big[M(T_u)\,\big|\,T_u \leq t_0\big]} \leq e^{-ru}\sup_{0\leq t\leq t_o} l(r;t)\ . \tag{40}$$

Taking expectations proves (24). Using (39) in (40) yields (25). $\square$

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